$PSL(2,q)$ ACTING TOTALLY IRREGULARLY ON FINITE PROJECTIVE PLANES

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TABLE OF CONTENTS

ACKNOWLEDGEMENTS ii

ABSTRACT v

CHAPTERS

1 INTRODUCTION 1

2 BACKGROUND 6

2.1 Geometry and Collineation Groups 6
2.2 Reflections of Conjugacy in Geometry 8
2.3 $PSL(2,q)$ 9
2.4 Known Results 11

3 SETTING UP THE INVOLUTION GEOMETRY 13

3.1 Some Group Results 13
3.2 Setting up Geometry Based on Involutions 17
  3.2.1 $C(i)$ and $a(i)$ 17
  3.2.2 D-centers and D-axes 21
  3.2.3 Stabilizer of a Point and a Line 28
  3.2.4 Intersections of D-axes 31
3.3 Theorems 36
  3.3.1 Bound for the Order of the Plane 36
  3.3.2 $PSL(2,q)$ Acts Strongly Irreducibly or Fixes a Subplane of Order q 38
  3.3.3 Assume No Point is Fixed by an Element of Order p 39

4 CONTRIBUTION OF ELEMENTS OF ORDER p 42

4.1 Base Lemmas 43
4.2 Triangular p-subgroups 47
4.3 $\mathcal{N}_{\mathcal{G}}(S)$ Fixes a Line or a Point 49
4.4 The p-element as Generalized Perspectivity 57
4.5 A Planar p-element 68
4.6 Proofs of Theorems 1.1 and 1.2 74

5 $PSL(2,7)$ 77

6 CONCLUSION 84

REFERENCES 86

BIOGRAPHICAL SKETCH 87

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$PSL(2,q)$ ACTING TOTALLY IRREGULARLY ON FINITE PROJECTIVE PLANES

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Assume that $PSL(2,q)$ acts totally irregularly on a finite projective plane Π of order n, that $PSL(2,q)$ contains an involutory homology, and that $q \equiv 3 \pmod 4$ where $q = p^r$ for prime p. Then $n < \frac{p}{p-1}(q^2-1)$. The cases where $n \leq q$ or $q = 3$ have been solved elsewhere; in this work there are results for the cases where $n > q$ and $p \neq 3$. It is shown that either the elements of order p are planar and $\mathcal{N}_{\mathcal{G}}(S)$ does not fix a line for S a Sylow p-group; the elements of order p are generalized elations, and $PSL(2,q)$ leaves invariant a Desarguesian subplane of order q in Π, a non-Desarguesian plane of order q^2; or $p = 3$ and the elements of order p are generalized elations; or the elements of order p act fixed-point-freely. In the case where elements of order p act fixed-point-freely, equations are found for n in terms of q. If, furthermore, the order of the stabilizer of every point is even, $q = 7$ and Π is a Desarguesian subplane of order 9 or 11.

Further results are that either $PSL(2,q)$ acts strongly irreducibly on the plane, or $PSL(2,q)$ leaves invariant a Desarguesian subplane of order q, $n = q^2$, and the elements of order p are generalized elations. If $PSL(2,q)$ leaves invariant a subplane of order q, then it is found that $n = q^2$ and the elements of order p are generalized elations.

Under the assumptions that $PSL(2,7)$ acts totally irregularly and that $PSL(2,7)$ contains a perspectivity, it is found that Π is a Desarguesian plane of order 2, 4, 7, 8, 9, 11, or n is 37, 39, or 49.

CHAPTER 1
INTRODUCTION

A basic question in the study of finite projective planes has been what types and what orders of finite projective planes exist. The Desarguesian plane is the most familiar; it is defined in terms of one- and two-dimensional subspaces of a three dimensional vector space over a finite field. The finite field is of prime power order and this order is also the order of the projective plane. The other projective planes which have been found are also of prime power order, and so a major conjecture in the theory of finite projective planes is that all finite projective planes are of prime power order.

One approach to finding new planes is to assume the existence of a particular group as a collineation group of a finite projective plane, and then see what must be true of the plane. The group imposes a structure on the plane, with some points and lines named in terms of the group, and intersections and incidences of the geometry reflected from relationships of the group. In this way, perhaps a new plane can be found which is based on the structure of the group, or else a new way can be found to think of the structure of an old plane. For instance, Lüneberg did this with $PSL(2,q)$ acting on a projective plane of order q. He determined that this plane must be Desarguesian (Result 2.2).

Just assuming the existence of a group acting on the plane may not give enough structure. So more restrictions on the type of action are introduced. For instance, Hering introduced the idea of *strongly irreducible* action, which means that no point, line, triangle, or subplane is left invariant by the group. If a non-abelian simple finite collineation group $\mathcal{G}$ contains a non-trivial perspectivity and acts strongly irreducibly, then Reifart and Stroth proved that $\mathcal{G}$ is isomorphic to one of the following: $PSL(2, q)$, $PSL(3, q)$, $PSU(3, q)$, A_7, or J_2 [15]. Hering proved that if $PSL(3, q)$ acts strongly irreducibly on a projective plane and contains a perspectivity, then the plane is Desarguesian of order q [5]. The known planes which admit a strongly irreducible subgroup with a perspectivity have been found to be Desarguesian.

Chat Yin Ho has introduced the idea of *totally irregular* action. Totally irregular of a group $\mathcal{G}$ on a plane Π means that every point in Π is fixed by a non-trivial subgroup of $\mathcal{G}$. This action occurs in non-Desarguesian as well as Desarguesian planes; all of the known planes admit a totally irregular collineation group. A totally irregular action forces an upper bound on the number of points as a function of the number of elements of the group since each element of the group fixes only a finite number of points. Ho and Gonçalves have looked at various groups acting totally irregularly where the groups contain a perspectivity. In particular, they look at $PSL(2, q)$, with $q \equiv 1 \pmod 4$, with further restrictions on the action of the group [10].

In this thesis it is assumed that $PSL(2, q)$ acts totally irregularly on a projective plane of order n, with $q \equiv 3 \pmod 4$ where $q = p^r$ for p prime. In general, the existence of an involutory homology is also assumed. $PSL(2, 7)$ is studied with less restriction; in this case the existence of a perspectivity is assumed.

The case where $n = q$ is known. Lüneberg proved that a plane of order q admitting $PSL(2, q)$ is Desarguesian (Result 2.2 in this paper); the Desarguesian plane of order q does admit $PSL(2, q)$ acting totally irregularly. Moorehouse has studied the cases where $n < q$, finding that the plane is Desarguesian and that $(n, q) = (2, 3), (2, 7), (4, 5), (4, 7)$ or $(4, 9)$ [14]. Note that in these cases the action of the group is necessarily totally irregular. Ho has shown that if $PSL(2, 3)$ acts totally irregularly on a projective plane of order n, then the plane is Desarguesian and n is 2, 3, or 4 [9]. Thus the cases left to be considered are where $n > q$ and $q \geq 7$.

Putting the totally irregular action together with the existence of an involutory homology further restricts the group-induced structure of the plane. The existence of one involutory homology forces all involutions to be homologies. The involutory centers and axes generate a substructure. Any two involutory centers are joined by an involutory axis or a type of line called a d-axis; any two involutory axes intersect in an involutory center or a d-center. The points fixed by non-trivial elements of order prime to p lie on the d-axes and involutory axes. The d-centers, involutory centers, involutory axes and d-axes provide a basic structure to the plane.

Next, the fixed points and lines of an element of order p are considered. Observing that all elements of order p have the same fixed point/line substructures, different possibilities for those substructures are studied. An element of order p may have no fixed points, or it may be triangular or planar, or be a generalized perspectivity.

The next two theorems summarize the major results of this paper.

Theorem 1.1 *Assume p is prime, q is a power of p where $q \equiv 3 \pmod 4$, and that $PSL(2,q)$ acts totally irregularly on a projective plane Π of order n. Assume the existence of an involutory homology. Let S be a Sylow p-subgroup of $PSL(2,q)$. The case where $q = 3$ occurs if and only if Π is a Desarguesian plane of order 3. For $q \geq 7$, $n < \frac{p}{p-1}(q^2-1) \leq \frac{3}{2}q^2$, and one of the following statements is true:*

1. *Either $n = q$ or $n = q^2$, and the elements of order p are generalized elations. The case where $n = q$ occurs if and only if Π is a Desarguesian plane of order q. If $n = q^2$, there is a Desarguesian subplane of order q left invariant by $PSL(2,q)$, and Π is not Desarguesian.*

2. *The elements of order p are generalized elations and $p = 3$. There is a common center for the elements of a Sylow p-group, and there are distinct axes for distinct subgroups of order p.*

3. *The stabilizer of every point is p-prime and $n = (\frac{q^2-1}{2}) \pm \sqrt{\frac{1}{8}(q^4 - 3q^3 + 9q^2 - q - 6)}$ or $n = (\frac{q^2-1}{2}) \pm \sqrt{\frac{1}{8}(q^4 - 5q^3 + 13q^2 + 5q - 6)}$. In particular, if the stabilizer of every point is of even order, then $q = 7$, and Π is the Desarguesian plane of order 9 or 11. The Desarguesian planes of order 9 and 11 do admit $PSL(2, 7)$ as a totally irregular collineation group with an involutory homology.*

4. *The element of order p is planar, q is not prime, and $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point but no line.*

5. *The element of order p is planar and $\mathcal{N}_{\mathcal{G}}(S)$ fixes no point and no line.*

No element of order p is triangular or a generalized homology. If $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line no element of order p is planar. If $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point then S is not planar.

The planes which have so far been found to satisfy the assumptions of this theorem are the Desarguesian planes of order q and the Desarguesian planes of order 9 and 11. There is a possibility of a plane of order q^2 with a Desarguesian subplane of order q. This plane is not Desarguesian.

The next theorem shows that for the group $PSL(2, q)$, totally irregular action implies strongly irreducible action except if $PSL(2, q)$ leaves invariant a subplane of order q.

Theorem 1.2 *Assume that $PSL(2, q)$ acts totally irregularly on the plane, with the same restrictions on q as above, and assume the existence of an involutory homology.*

$PSL(2,q)$ either acts strongly irreducibly on the plane or leaves invariant a Desarguesian subplane of order q. If $PSL(2,q)$ leaves invariant a Desarguesian subplane of order q, then $n = q^2$ and the element of order p is a generalized elation.

In particular, no point or line is left invariant by $PSL(2,q)$. The only known projective planes where the collineation group does not fix a point and a line are the Desarguesian, Hughes, and Figueroa planes.

The case where $q = 7$ is studied in greater generality through the assumption of the existence of a perspectivity rather than an involutory homology. $PSL(2,7)$ is in some sense "special" in the class of subgroups $PSL(2,q)$ with $q \equiv 3 \pmod 4$. It is a subgroup of $PSL(3,q)$ when $q^3 \equiv 1 \pmod 7$, and so is a collineation group of an infinite number of Desarguesian planes. With the added assumption of a totally irregular action, $PSL(2,7)$ is the collineation group of only a finite number of planes. However it is still unique in the context of the assumptions of this thesis: it is the only collineation group where the stabilizer of every point is even.

Theorem 1.3 *Assume $\mathcal{G} \cong PSL(2,7)$ is a totally irregular collineation group of a finite projective plane Π of order n, and assume $\mathcal{G}$ contains a perspectivity. Then an involution is either a homology or an elation. If an involution is a homology, then either n is 7, 9, 11, 37, or 39 and the element of order 7 fixes no points; or n is 49 and the element of order 7 is a generalized elation. The cases where n is 7, 9 or 11 do occur, and the plane is Desarguesian. If the involution is an elation, then n is 2, 4, or 8, and Π is Desarguesian. The Desarguesian planes of order 2, 4, and 8 do admit $PSL(2,7)$ as a totally irregular collineation group with an involutory elation.*

CHAPTER 2
BACKGROUND

Some basic definitions and properties from finite geometry are accumulated here for later reference. Properties of the group $PSL(2, q)$ and a collection of other results used in this paper are listed.

2.1 Geometry and Collineation Groups

A *closed configuration* is a substructure of points and lines where any two points meet in a unique line of the set and any two lines meet in a unique point of the set. These conditions are two out of the three axioms for a projective plane. If the substructure also has four points, no three collinear, then it is a projective plane. Otherwise the closed configuration is called a *degenerate projective plane* [11]. A degenerate projective plane consists of points and lines where all but possibly one of the points lie on one of the lines, and all but possibly one of the lines is incident to one of the points. It is possible to have just one point or just one line in the degenerate plane.

If a subplane Π_o of a projective plane Π has the property that every line of Π is incident to a point of Π_o and every point of Π is on a line of Π_o, then Π_o is called a *Baer* subplane.

A *collineation* of a projective plane is an isomorphism of the plane onto itself, taking points to points, lines to lines, and preserving incidence. The set of all collineations forms a group called the (full) collineation group of the plane. A subgroup of the full collineation group is termed a collineation group. (page 90, [11].)

For a subset $\mathcal{K}$ of the group, let $\mathcal{P}(\mathcal{K})$ (respectively $\mathcal{L}(\mathcal{K})$) denote the set of fixed points (lines) of $\mathcal{K}$, $\mathcal{F}(\mathcal{K})$ the set of points and lines fixed by $\mathcal{K}$. The fixed points and lines form a closed configuration: the line through any two fixed points is also fixed, and the point incident to two fixed lines is also fixed. So the substructure $\mathcal{F}(\mathcal{K})$ is either a subplane or a degenerate subplane.

An element x of a collineation group is called *planar* if $\mathcal{F}(x)$ is a subplane, and *triangular* if $\mathcal{F}(x)$ is a triangle. If x fixes all of the points on a line $\mathcal{L}$ and hence $\mathcal{L}$ itself, it is called a *perspectivity.* $\mathcal{L}$ is called the axis of the perspectivity and denoted $a(x)$. All of the lines through a point are also fixed by x, and that point is called the center of the perspectivity and is denoted $C(x)$. If $C(x)$ is incident to $a(x)$, then x is called an *elation*; otherwise it is called a *homology.*

If x has exactly one fixed point and one fixed line, then x is called *flag* if the point and line are incident, *antiflag* otherwise. If x has fixed points, all but possibly one of which lie on a line $\mathcal{L}$, then x is called a *generalized* perspectivity. If all of its fixed points lie on $\mathcal{L}$, it is called a *generalized elation*; if there exists a fixed point off the line it is called a *generalized homology.* So flag, antiflag, and triangular collineations can also be considered generalized perspectivities. For a non-triangular generalized perspectivity x, the line $\mathcal{L}$ is unique and will be called a *generalized axis* $a(x)$. The fixed lines of x except possibly $a(x)$ will be incident to a point which will be called a *generalized center* $C(x)$. In the case that x is triangular, the three lines are indistinguishable; no line will be denoted $a(x)$, no point, $C(x)$.

These definitions are generalized to subgroups of the collineation group. So, for instance, a subgroup H is called planar if its fixed points and lines form a subplane. Otherwise its fixed points and lines form a degenerate projective plane. If there is only one fixed point, or if there are more than two fixed lines of H through a point, then that point is unique and will be denoted $C(H)$. Similarly, if there is only one fixed line, or more than two fixed points of H incident to a line, that line will be denoted $a(H)$. The existence of one does not imply the other.

2.2 Reflections of Conjugacy in Geometry

A group acts on itself through conjugation. This action is reflected in the action of the group on the finite plane. Let x, y and g be elements of $\mathcal{G}$. If $x \xrightarrow{g} y$, in other words $g^{-1}xg = y$, then g maps the fixed points and lines of x to the fixed points and lines of y. For if P is a fixed point of x, then $P^x = P$, and

$$P^g = P^{xg} = P^{(gg^{-1})xg} = P^{g(g^{-1}xg)} = P^{gy} = (P^g)^y.$$

So $P^g = (P^g)^y$, and P^g is a fixed point of y. This same argument works for a fixed line, a set of points left invariant by x, in fact any structure left invariant by x. So a substructure left invariant by x is taken by g to an isomorphic substructure left invariant by x^g.

This means that all elements in the conjugacy class of x have isomorphic fixed point/line substructures. So knowing the fixed point substructure of an element x means knowing the structures for the whole conjugacy class of x.

If g commutes with x then g takes x to x through conjugacy and hence g permutes the fixed points and the fixed lines of x. If x is a nontriangular generalized perspectivity then g will fix $C(x)$ since it is unique among the fixed points of x. Similarly, g will fix $a(x)$.

The same results are true for the fixed points and lines of a subgroup H. If $H \xrightarrow{g} K$, in other words $g^{-1}Hg = K$, then g maps the fixed points and lines of H to the fixed points and lines of K. So conjugates have isomorphic fixed substructures, and if an element g normalizes H then g fixes H and permutes the fixed points and lines of H. If there is a unique point such as a generalized center, then it will be fixed by g.

These observations will be put together for later reference.

Result 2.1 *If* $x^g = y$, *then* $\mathcal{F}(x)^g = \mathcal{F}(y)$ *for* x, y *elements of* $\mathcal{G}$. *If there is a generalized center* $C(x)$ *or axis* $a(x)$, *then* $C(x^g) = C(x)^g$, $a(x^g) = a(x)^g$. *If* $H \leq \mathcal{G}$, $\mathcal{F}(H^g) = (\mathcal{F}(H))^g$. *If there is a generalized center* $C(H)$ *or axis* $a(H)$, *then* $C(H^g) = C(H)^g$ *and* $a(H^g) = a(H)^g$.

2.3 $PSL(2,q)$

In the following, assume that $q \equiv 3 \pmod 4$, and that $q = p^r$ for prime p.

The group $PSL(2,q)$ has been studied in many sources, such as [3,4,12,16]. For the purposes of this paper, the major reference is Dickson [3]. There are many ways to look at $PSL(2,q)$. It is defined as the factor group $SL(2,q)/Z_1$(2,q), where Z_1(2,q) is the center of $SL(2,q)$ (page 166, [16]). The center of $SL(2,q)$ consists of the scalar matrices with determinant 1. In geometric terms, $PSL(2,q)$ is induced by the action of $SL(2,q)$ on the projective line over the field with q elements. Many of the properties of $PSL(2,q)$ are derived from this group action. It is convenient in studying this action to name the points of the line in parametric coordinates, and to associate $PSL(2,q)$ with the linear fractional mappings $x \hookrightarrow \frac{ax+b}{cx+d}$ where $ad - bc$ is a non-zero square (see, for instance, Lemma 2.14, page 34 of "Projective Planes" [11]). Dickson uses this correspondence in his chapter on $PSL(2,q)$, chapter XII of "Linear Groups" (see pages 260-287 of "Linear Groups" [3]).

Here is an accumulation of his results.

Result 2.2 1. *$PSL(2,q)$ contains exactly $\frac{1}{2}q(q+1)$ conjugate cyclic subgroups of order $(q-1)/2$, exactly $\frac{1}{2}q(q-1)$ conjugate cyclic subgroups of order $(q+1)/2$ and exactly $q+1$ conjugate elementary abelian p-groups of order q (section 260).*

2. *The set of all these groups is a partition of* $PSL(2,q)$ *(sections 241, 242, 243 show these groups have trivial intersection, section 244 shows all elements are accounted for in these groups).*

3. *The normalizer of a cyclic group of order* $(q-1)/2$ *is a dihedral group of order* $q-1$*, of a cyclic group of order* $(q+1)/2$ *is a dihedral group of order* $q+1$*, and of an elementary abelian p-group of order* q *is a semi-direct product of order* $q(q-1)/2$ *(sections 241, 242, and 243).*

4. *A cyclic group of order* $(q-1)/2$ *is in precisely two normalizers of elementary abelian p-groups of order* q*. A normalizer of an elementary abelian p-group of order* q *is characterized by being the stabilizer of a point on the projective line (section 241). A cyclic group of order* $(q-1)/2$ *fixes exactly two points of the projective line (section 242).*

5. *If a subgroup* H *of* $PSL(2,q)$ *contains an element of order* p *and* r *is odd, then* H *is either* $PSL(2,p^k)$ *where* $1 \leq k \leq r$*, or a subgroup of* $\mathcal{N}_{\mathcal{G}}(S)$*, where* S *is an elementary abelian p-group of order* q*. (sections 249, 250, and 251).*

6. $PSL(2,q)$ *is simple for* $q > 3$ *(section 261).*

7. *The elements of order* p *in a Sylow p-group* S *separate into two conjugacy classes, and the* $p-1$ *elements of a cyclic group of order* p *belong half to one and half to another if* r *is odd. (section 241).*

8. *If* $H \leq S$ *and* $|H| = p^m$*, then* $\mathcal{N}_{\mathcal{G}}(H)$ *is a a semi-direct product of* S *with a cyclic group of order* $t = (p^d - 1)/2$*, where* $d|(m,r)$ *(sections 249 and 250).*

The normalizer of a Sylow p-group S, the semidirect product of S with a cyclic group of order $\frac{1}{2}(q-1)$, is a Frobenius group (See, for instance, page 41 of "Finite

Groups" [4]). A subgroup of $\mathcal{N}_{\mathcal{G}}(S)$ which is the product of a non-trivial group $H \leq S$ with a non-trivial cyclic group of order t where $t \mid (|H|-1)$ is also a Frobenius group.

Dickson has a summary of the subgroups of $PSL(2,q)$. I use Lüneberg's listing from pages 61 and 62 of "Translation Planes" with a slight reformulation reflecting the restriction that $q \equiv 3 \pmod 4$.

Result 2.3 *If U is a subgroup of PSL(2,p^r), then U is one of the following groups:*

1. *An elementary abelian p-group of order p^m with $m \leq r$.*
2. *A cyclic group of order z where z is a divisor of $\frac{1}{2}(p^r-1)$ or $\frac{1}{2}(p^r+1)$ for p odd.*
3. *A dihedral group of order $2z$ where z is as in 2.*
4. *A semidirect product of an elementary abelian p-group of order p^m and a cyclic group of order t where t is a divisor of $(p^{(m,r)}-1)/2$. [This is a reformulation of Lüneberg's statement]*
5. *A group isomorphic to A_4.*
6. *A group isomorphic to S_4. In this case $p^{2r}-1 \equiv 0 \pmod{16}$.*
7. *A group isomorphic to A_5. In this case $p^r(p^{2r}-1) \equiv 0 \pmod 5$.*
8. *A group isomorphic to PSL(2,p^m) where m divides r.*
9. *A group isomorphic to PGL(2,p^m) with $2m$ a divisor of r.*

2.4 Known Results

Result 2.4 *A collineation of a finite projective plane has an equal number of fixed points and lines. If M is a cyclic collineation group of a finite projective plane Π, then the point orbits of Π under M have the same cyclic structure as the line orbits.*

See page 256 in "Projective Planes" [11].

Lemma 2.1 *Suppose x and y commute. If x and y are generalized elations, then either $C(x) = C(y)$ or $a(x) = a(y)$. If x and y are non-triangular generalized homologies, then $C(x) = C(y)$ if and only if $a(x) = a(y)$.*

Theorem 2.1 *Let α and β be two perspectivities of $\mathcal{G}$ and let $\sigma = \alpha\beta$. Then $\alpha\beta$ is a generalized perspectivity or the identity. If $C(\alpha) \neq C(\beta)$ and $a(\alpha) \neq a(\beta)$, then σ fixes $(a(\alpha) \cap a(\beta))$, and any remaining fixed points of σ lie on $\overline{C(\alpha)C(\beta)}$. Similarily, σ fixes $\overline{C(\alpha)C(\beta)}$, and any other fixed lines of σ are incident to $(a(\alpha) \cap a(\beta))$.*

See Lemma 5.1 [6].

Theorem 2.2 *Let $p \geq 3$ be a prime, and let q be a power of p. If Π is a projective plane of order q admitting a collineation group isomorphic to $PSL(2, q)$, then Π is Desarguesian.*

See page 199 in "Translation Planes" [13].

Theorem 2.3 *Let Π be a finite projective plane of order n with a proper subplane Π_o of order m. Then either $n = m^2$, or $n \geq m^2 + m$. Π_o is a Baer subplane if and only if $n = m^2$.*

See, for instance, pages 81-82 in "Projective Planes" [11].

Theorem 2.4 *Let $\mathcal{G} \cong PSL(2, q)$, $q > 3$ a prime power, be a collineation group of a finite projective plane containing an involutory homology. If $\mathcal{G}$ does not have any regular orbit of points, then $\mathcal{G}$ does not fix any point or line.*

See Theorem A in "On $PSL(2, q)$ as totally irregular collineation groups" [10].

CHAPTER 3
SETTING UP THE INVOLUTION GEOMETRY

In this thesis, assume that $\mathcal{G} \cong PSL(2,q)$ where $q \equiv 3 \pmod 4$ and $q = p^r$ for p prime, r an integer. Assume furthermore that $\mathcal{G}$ acts totally irregularly on a projective plane of order n, and in chapters 3 and 4, that $\mathcal{G}$ contains an involutory homology. The case where $q = 3$ has been determined in "Totally Irregular Collineation Groups and Finite Desarguesian Planes" by Chat Yin Ho [9]. The case where $n < q$ has been studied in "$PSL(2,q)$ as a Collineation Group of Projective Planes of Small Order" [14], and the case where $n = q$ has been solved by Lüneberg [13]. So, in chapters 3 and 4, it will be assumed that $q \geq 7$ and $n > q$.

First, some observations about the subgroup structure of $\mathcal{G}$ are made. Then the existence of an involutory homology is used to set up some structure of the plane. Points are separated into disjoint sets. The geometry of the plane is used to get a rough bound on the order of the plane, and to show that either $PSL(2,q)$ acts strongly irreducibly on Π or leaves invariant a Desarguesian subplane of order q. Another result based on the established structure gives values of n in terms of q when it is assumed that no element of order p fixes a point. In particular if the stabilizer of every point is of even order, $q = 7$, and Π is the Desarguesian plane of order 9 or 11.

3.1 Some Group Results

Lemma 3.1 *If* $q \equiv 3 \pmod 4$ *then* $p \equiv 3 \pmod 4$ *and* r *is odd. Thus there are no subgroups in* $\mathcal{G}$ *of the form* $PGL(2,p^m)$.

Proof: The proof of the equivalence is clear by computing.

The list of subgroups given in result 2.3 shows that for $PGL(2,p^m)$ to be a subgroup of $PSL(2,q)$, $2m$ must be a divisor of r.

□

Lemma 3.2 1. *A subgroup of $\mathcal{G}$ isomorphic to the dihedral group D_{q+1} is maximal for $q > 7$.*

2. *If i is an involution then $C_{\mathcal{G}}(i) \cong D_{q+1}$.*
3. *There are $\frac{q+1}{2}$ non-central involutions in $C_{\mathcal{G}}(i)$ which are partitioned into commuting pairs whose product is i. Each non-central involution commutes with exactly two involutions in $C_{\mathcal{G}}(i)$. There are $\frac{q+1}{4}$ subgroups isomorphic to the Klein 4 group in $C_{\mathcal{G}}(i)$.*
4. *An involution is central in exactly one dihedral subgroup of order $q+1$ and is non-central in exactly $\frac{q+1}{2}$ other dihedral subgroups of order $q+1$.*
5. *Two dihedral subgroups of $\mathcal{G}$ of order $q+1$ intersect in a Klein 4 group, a group of order 2, or in the identity.*
6. *All dihedral groups of order $q+1$ are conjugate.*

Proof: For part (1), assume that a subgroup H of $\mathcal{G}$ is isomorphic to D_{q+1}. The list of subgroups of $\mathcal{G}$ in result 2.3 shows that the subgroups that could contain H are ones that are isomorphic to A_4, S_4, or A_5. Since the assumption in (1) is that $q > 7$, and since $q \equiv 3 \pmod 4$, $q \geq 11$. There then exists a cyclic subgroup of order $(q+1)/2 \geq 6$ in H. This fact rules out A_4, S_4 and A_5 since the orders of their elements are at most 5.

Now look at assertion (2). The partition property states that every element is in either a cyclic group of order $\frac{q+1}{2}$ or $\frac{q-1}{2}$ or in a Sylow p-group. Since $q \equiv 3 \pmod 4$, $\frac{q+1}{2}$ is the only choice that is even. Each cyclic group H of order $\frac{q+1}{2}$ has precisely one involution i. H is normalized by a dihedral group M of order $q+1$ by result 2.2, and so $M \leq \mathcal{C}_{\mathcal{G}}(i)$. For $q > 7$ the dihedral group of order $q+1$ is maximal and so $M = \mathcal{C}_{\mathcal{G}}(i)$. If $q = 7$, M is contained in a group isomorphic to S_4 which does not centralize any involution, and so $M = \mathcal{C}_{\mathcal{G}}(i)$.

Part (3) follows from looking at the internal structure of a dihedral group. Since $q \equiv 3 \pmod 4$, $\frac{q+1}{2}$ is even and the statements follow from knowledge of dihedral groups.

The statement of part (4) follows by looking at $\mathcal{C}_{\mathcal{G}}(i)$. Since i commutes with the $\frac{q+1}{2}$ involutions in $\mathcal{C}_{\mathcal{G}}(i)$, i in turn will be in each of their centralizers, putting it in $\frac{q+1}{2}$ more dihedral groups of order $q+1$.

For part (5) let i and j be two involutions. Suppose $\mathcal{C}_{\mathcal{G}}(i) \cap \mathcal{C}_{\mathcal{G}}(j) = H$. $\mathcal{C}_{\mathcal{G}}(i)$ and $\mathcal{C}_{\mathcal{G}}(j)$ are normalizers of distinct cyclic subgroups of order $\frac{q+1}{2}$, cyclic groups which do not intersect by the partition property of $PSL(2,q)$. Since the dihedral group of order $q+1$ is made up of this cyclic part together with $\frac{q+1}{2}$ involutions, the two dihedral groups can only intersect in involutions. If i and j commute, then $\mathcal{C}_{\mathcal{G}}(i) \cap \mathcal{C}_{\mathcal{G}}(j) = \langle i, j \rangle$ is a Klein 4 group. Otherwise the intersection is a group of order 2 or the identity.

Part (6) is a consequence of result 2.2. The cyclic subgroups of order $\frac{q+1}{2}$ are conjugate, and hence the normalizers of these groups are.

□

Lemma 3.3 *1. The dihedral subgroups of order* $q-1$ *are conjugate.*

2. *A subgroup H isomorphic to the dihedral group of order $q-1$ is maximal for $q > 11$. If $q = 11$, then H is contained in a subgroup isomorphic to A_5.*

3. *There are $\frac{q-1}{2}$ involutions in H together with an odd cyclic group of order $\frac{q-1}{2}$.*

4. *Two distinct groups isomorphic to dihedral groups of order $q-1$ intersect in either a group of order 2 or 1.*

5. *If $K \cong D_{q+1}$ and $L \cong D_{q-1}$, then K intersects L in a subgroup of order 2 or 1.*

Proof:

Result 2.2 states that the normalizer of a cyclic group of order $\frac{q-1}{2}$ is a dihedral group of order $q-1$ and that the cyclic groups of order $\frac{q-1}{2}$ are conjugate. Part (1) then follows since normalizers of conjugate groups are conjugate.

For part (2), order considerations eliminate all of the groups except possibly A_4, S_4, and A_5. If $q > 11$ then $q \geq 19$ and $\frac{q-1}{2} \geq 9$, and so there is an element of order $\frac{q-1}{2} \geq 9$ in H. Since the order of any element in A_4, S_4, or A_5 is at most 5, H is maximal for the case $q > 11$.

Now suppose that $q = 11$. H is then a dihedral group of order 10. $PSL(2,11)$ contains subgroups isomorphic to A_5, and A_5 contains dihedral groups of order 10. Since all of the dihedral groups are conjugate there must be some group isomorphic to A_5 which contains H.

Part (3) follows by noting that since $q \equiv 3 \pmod 4$, $\frac{q-1}{2}$ is odd.

For part (4) suppose that K and L are distinct groups isomorphic to D_{q-1}. Then each consists of a cyclic group of order $\frac{q-1}{2}$ together with $\frac{q-1}{2}$ other elements which are involutions. The cyclic groups have trivial intersection by the partition property for $PSL(2,q)$, so K and L intersect in involutions or the identity. If they intersect in two or more involutions, a dihedral group of order $t > 2$ where $2|t$ and $t|(q-1)$

is generated in the intersection. But this contradicts the fact that cyclic groups of order $\frac{q-1}{2}$ have trivial intersection.

For part (5) suppose $K \cong D_{q+1}$ and $L \cong D_{q-1}$. The cyclic parts of the two groups intersect trivially by order considerations. There could be a common involution, but if there were two or more they would generate a dihedral group which would have a non-trivial cyclic part common to both K and L.

□

Lemma 3.4 *If a subgroup* H *of* $PSL(2,q)$ *contains an element of order* p, *then either* $H = PSL(2,p^k)$ *for* $k \leq r$ *or* $H \leq \mathcal{N}_{\mathcal{G}}(S)$ *for some Sylow p-group* S. $\mathcal{N}_{\mathcal{G}}(S)$ *is maximal for* S *a Sylow p-group of* $PSL(2,q)$. *If* $H^g = \bar{H}$ *for* H, $\bar{H}$ *subgroups of* S, *then* $g \in \mathcal{N}_{\mathcal{G}}(S)$.

Proof: By result 2.2 if H contains an element of order p, then either r is even where $q = p^r$, $H = PSL(2,p^k)$, or $H \leq \mathcal{N}_{\mathcal{G}}(S)$ for some Sylow p-group S. By Lemma 3.1, r is odd, so $H = PSL(2,p^k)$ or $H \leq \mathcal{N}_{\mathcal{G}}(S)$.

The maximality of $\mathcal{N}_{\mathcal{G}}(S)$ follows from the first assertion of Lemma 3.4.

Suppose that $H^g = \bar{H}$ for H and $\bar{H}$ subgroups of S and $g \in \mathcal{G}$. Since distinct Sylow p-groups intersect in the identity by the partition property of $PSL(2,q)$, either $S = S^g$ or $S \cap S^g = \langle 1 \rangle$. Since both H and H^g are in S, this fact forces $S = S^g$. Thus $g \in \mathcal{N}_{\mathcal{G}}(S)$.

□

3.2 Setting up Geometry Based on Involutions

3.2.1 $C(i)$ and $a(i)$

The assumption of the existence of an involutory homology is used to set up a base substructure in the projective plane. Each involution is a homology with an associated center and axis. The stabilizer of the center and axis is $\mathcal{C}_{\mathcal{G}}(i)$. From this fact it is seen that distinct involutions have distinct centers and axes. It follows that the fixed points of any element in a dihedral group of order $q+1$ lie on an involutory axis. For $i \neq j$, the involutory center $C(i)$ lies on an involutory axis $a(j)$ if and only if i and j commute. Thus the number of involutory centers on an involutory axis, and dually, the number of involutory axes incident to an involutory center is established.

Lemma 3.5 *There are $\frac{1}{2}q(q-1)$ involutions, all of which are conjugate. If one of them is a homology then all of them are.*

Proof: Let i be an involutory homology. By the partition property for $PSL(2,q)$, every element is in precisely one of the following: a cyclic subgroup of order $\frac{1}{2}(q \pm 1)$ or an elementary abelian group of order q. Since $q \equiv 3 \bmod 4$, the involutions must be in the cyclic groups of order $\frac{q+1}{2}$. There are $\frac{1}{2}q(q-1)$ of these subgroups, each with precisely one involution, so there are $\frac{1}{2}q(q-1)$ involutions. Since the subgroups are conjugate, so are the involutions.

Since all of the involutions are conjugate, they have the same fixed point/line structure. So the existence of one involutory homology forces all of the involutions to be homologies.

□

Lemma 3.6 *For involutory center $C(i)$ and axis $a(i)$, $\mathcal{G}_{C(i)} = \mathcal{C}_{\mathcal{G}}(i)$ and $\mathcal{G}_{a(i)} = \mathcal{C}_{\mathcal{G}}(i)$. Furthermore, distinct involutions have distinct centers and distinct axes.*

Proof:

By Result 2.1, if $x \in \mathcal{C}_{\mathcal{G}}(i)$, then x fixes $C(i)$. Thus $\mathcal{C}_{\mathcal{G}}(i) \leq \mathcal{G}_{C(i)}$. If $q > 7$, then Lemma 3.2 shows that $\mathcal{C}_{\mathcal{G}}(i)$ is maximal. Since Theorem 2.4 shows that $\mathcal{G}$ does not fix any point, we have $\mathcal{G}_{C(i)} = \mathcal{C}_{\mathcal{G}}(i)$. This then shows that distinct involutions have distinct centers. Similarly, $\mathcal{G}_{a(i)} = \mathcal{C}_{\mathcal{G}}(i)$ and the involutory axes are distinct for the case $q > 7$.

Now assume $q = 7$. In this case $\mathcal{C}_{\mathcal{G}}(i)$ is not maximal; it is maximal in a subgroup isomorphic to S_4 which is maximal in $\mathcal{G}$, and so it is possible that $\mathcal{G}_{C(i)} \neq \mathcal{C}_{\mathcal{G}}(i)$.

Let $H = \mathcal{G}_{C(i)}$ and suppose $\mathcal{C}_{\mathcal{G}}(i) \neq H$. By Theorem 2.4, $H \neq \mathcal{G}$, so $H \cong S_4$. Let K be the Klein 4 subgroup of H which is normalized by H. Then $i \in K$, and let j be another involution in K. An element x of order 3 in H permutes the involutions i, j and ij, and hence their corresponding centers. Since x fixes $C(i)$, $C(i) = C(j) = C(ij)$. By Lemma 2.1 this means $a(i) = a(j) = a(ij)$ too, and so the involutions in K have a common center and common axis.

There are 7 distinct subgroups in the conjugacy class of K (see, for instance, section 260 of Dickson [3]). There are exactly 21 conjugate involutions, so each involution will be in exactly one of the subgroups. By the preceding paragraph, the three involutions in K^g have a common center and a common axis. The intersection of any two distinct conjugates of K is trivial, so if any two distinct subgroups in the conjugacy class of K share a common center C, that center will be common for 6 involutions. $\mathcal{G}_C$ then contains the centralizers of 6 involutions forcing $\mathcal{G}_C = \mathcal{G}$. Thus $\mathcal{G}_C = \mathcal{G}_{C(i)} = \mathcal{G}$ which is a contradiction. Similarily, no two distinct conjugates of K share a common axis. Thus the 7 distinct conjugates of K have distinct centers and distinct axes, each center and axis common for exactly three commuting involutions.

Now look again at H. By referring to the structure of S_4, it is seen that H contains 3 dihedral groups of order 8: $\mathcal{C}_{\mathcal{G}}(i)$, $\mathcal{C}_{\mathcal{G}}(j)$, and $\mathcal{C}_{\mathcal{G}}(ij)$. The 6 involutions in $H \backslash K$ are partitioned into these three dihedral groups, two in $\mathcal{C}_{\mathcal{G}}(i)$, two in $\mathcal{C}_{\mathcal{G}}(j)$, and two in $\mathcal{C}_{\mathcal{G}}(ij)$. Now it will be shown that these 6 involutions have 6 distinct centers. Let m and l be two of them, where $m \neq l$, and suppose $C(m) = C(l)$. Suppose first that m and l are in the same centralizer of either i, j or ij. Without loss of generality, assume m and l are in $\mathcal{C}_{\mathcal{G}}(i)$. Then $ml = i$, and since $C(m) = C(l)$, $C(m) = C(l) = C(i)$. However, m and l are not in K, and i shares a common center only with the involutions in K, so this contradiction proves $C(m) \neq C(l)$. Now suppose l and m are in different centralizers; without loss of generality, suppose $m \in \mathcal{C}_{\mathcal{G}}(i)$, $l \in \mathcal{C}_{\mathcal{G}}(j)$. Then, by referring to the structure of S_4, l and m do not commute, which contradicts the fact that a center is common for two involutions only if they commute.

So the 6 involutions of $H \backslash K$ have distinct centers on $a(i)$, and hence there are 6 involutory centers on each of the 7 involutory axes. This contradicts the fact that there are only 7 centers.

This contradiction proves $\mathcal{G}_{C(i)} = \mathcal{C}_{\mathcal{G}}(i)$, and hence distinct involutions have distinct centers. Similarly, $\mathcal{G}_{a(i)} = \mathcal{C}_{\mathcal{G}}(i)$, and distinct involutions have distinct axes.

□

Lemma 3.7 *Let $H \neq \langle 1 \rangle$ where H is a cyclic group of order t, $t | \frac{q+1}{2}$. Then $H \leq \mathcal{C}_{\mathcal{G}}(i)$ for some involution i, and H fixes $C(i)$ and $a(i)$. Any other fixed points of H are incident to $a(i)$; any other fixed lines of H are incident to $C(i)$.*

Proof: By the partition property for $PSL(2, q)$, H is a subgroup of precisely one dihedral group of order $q+1$ which, by Lemma 3.2, equals $\mathcal{C}_{\mathcal{G}}(i)$ for some involution i. Let x be a non-trivial element of H. Then x is the product of involutions k and j, where $k \neq i$ and $j \neq i$. These involutions are perspectivities with distinct centers and axes, so Theorem 2.1 gives that x is a generalized perspectivity which fixes $(a(k) \cap a(j))$ and whose other fixed points lie on $\overline{C(j)C(k)}$. Any fixed lines of x other than $\overline{C(j)C(k)}$ are incident to $a(k) \cap a(j)$. Since j and k both commute with i, $C(i) = a(k) \cap a(j)$ and $a(i) = \overline{C(j)C(k)}$. Thus x fixes $C(i)$, and any other fixed points are incident to $a(i)$; x fixes $a(i)$ and any other fixed lines are incident with $C(i)$.

Lemma 3.8 *Let i and j be distinct involutions. $C(i)$ I $a(j)$ if and only if i commutes with j. There are exactly $\frac{q+1}{2}$ involutory centers incident to $a(i)$ and exactly $\frac{q+1}{2}$ involutory axes incident to $C(i)$. If i commutes with j then $a(i)$ intersects $a(j)$ in $C(ij)$, $\overline{C(i)C(j)} = a(ij)$.*

Proof: If $C(i)$ I $a(j)$ then j fixes $C(i)$ which means $j \in \mathcal{C}_{\mathcal{G}}(i)$ by Lemma 3.6. If i commutes with j, then $j \in \mathcal{C}_{\mathcal{G}}(i)$ which means j fixes $C(i)$. Since $C(j) \neq C(i)$ by Lemma 3.6, $C(i)$ I $a(j)$.

There are $\frac{q+1}{2}$ involutions other than i in $\mathcal{C}_{\mathcal{G}}(i)$, so their corresponding centers are incident to $a(i)$ and their corresponding axes are incident with $C(i)$.

If i commutes with j, then ij is an involution, both i and j are in $\mathcal{C}_{\mathcal{G}}(ij)$, and hence both fix $C(ij)$ by Lemma 3.6. Since the involutory centers are distinct, $C(ij)$ is incident to both $a(i)$ and $a(j)$. Similarly, $a(ij)$ is incident to both $C(i)$ and $C(j)$.

$\square$

3.2.2 D-centers and D-axes

By looking at the centralizers of involutions, some structure of the plane has been established. Now some structure due to the dihedral group of order $q-1$ is presented. Each dihedral group of order $q-1$ fixes a unique point (d-center) and a unique line (d-axis). The geometry of the involutory centers, d-centers, involutory axes, and d-axes is explored. In particular, it is shown that two involutory axes intersect in either an involutory center or a d-center, and that involutory centers are joined by either an involutory axis or a d-axis.

Lemma 3.9 *For a subgroup $D \cong \mathcal{D}_{q-1}$ there is a unique point $C(D)$ and unique line $a(D)$ fixed by D. $\mathcal{G}_{C(D)} = D$ and $\mathcal{G}_{a(D)} = D$. Furthermore, any two dihedral groups of order $q-1$ have distinct fixed points and distinct fixed lines.*

Proof: Let i and j be involutory generators of D so $D = \langle i, j \rangle$. The point $P = a(i) \cap a(j)$ is fixed by both i and j so $D \leq \mathcal{G}_P$ and P is incident to all of the involutory axes of the involutions in D. By Lemma 3.6, P is not an involutory center since the stabilizer of an involutory center does not contain a dihedral group of order $q-1$. For $q > 11$, D_{q-1} is maximal, so since $\mathcal{G}_P \neq \mathcal{G}$ by Theorem 2.4, $\mathcal{G}_P = D$ when $q > 11$.

The cases where $q = 7$ and 11 are more difficult. Again, $\mathcal{G}_P \neq \mathcal{G}$. When $q = 7$ the only proper subgroup of $PSL(2,7)$ which contains D_6 is a group isomorphic to S_4; when $q = 11$ the only proper subgroup of $PSL(2,11)$ which contains D_{10} is a group isomorphic to A_5. So it is conceivable that in the case $q = 7$, $\mathcal{G}_P \cong S_4$, and in the case $q = 11$, $\mathcal{G}_P \cong A_5$.

In both cases, since P is not an involutory center, P is incident to $a(i)$ for an involution i in $\mathcal{G}_P$. Thus the involutory axes corresponding to involutions in $\mathcal{G}_P$ intersect in P. However, this cannot be since there are commuting involutions in $\mathcal{G}_P$. Lemma 3.8 shows two commuting involutions have their corresponding axes intersect in an involutory center. This contradiction proves $\mathcal{G}_P = D$.

If D fixes another point $Q \neq P$, then since Q is not an involutory center, all of the involutory axes corresponding to involutions in D must be incident to Q as well as to P. This contradiction proves that D fixes a unique point which will be called $C(D)$. For distinct dihedral subgroups D and $\tilde{D}$ of order $q-1$, $C(D) \neq C(\tilde{D})$.

Similarly, it is proved that $\overline{C(i)C(j)}$ is the unique line $a(D)$ fixed by D, and D is its stabilizer; distinct dihedral groups of order $q-1$ fix distinct lines.

□

Definition 3.1 *Denote a group isomorphic to $\mathcal{D}_{q-1}$ a* d-group, *using the symbol D as a general name for the group and $K(D)$ for the cyclic maximal normal subgroup of D. Let the point $C(D)$ and line $a(D)$ be called a* d-center *and a* d-axis, *respectively.*

Lemma 3.10 *No involutory axis is a d-axis; no involutory center is a d-center.*

Proof: If $a(i) = a(D)$ for an involution i and D a d-group, then $D \leq \mathcal{C}_{\mathcal{G}}(i)$ contradicting Lemma 3.6.

Similarly, the centers are distinct.

□

Lemma 3.11 *Let H be a non-trivial cyclic group of order t where $t|\frac{q-1}{2}$. Then $H \leq D$ for exactly one d-group D, H fixes $C(D)$ and $a(D)$, and any other fixed points of H lie on $a(D)$, any other fixed lines of H are through $C(D)$.*

Proof: By the partition property of $PSL(2,q)$ (Result 2.2), $H \leq K(D)$ for exactly one d-group D. Let $1 \neq x \in H$. Since $x \in D$, x is the product of involutions i and j in D, and so by Theorem 2.1, x is a generalized perspectivity which fixes $a(i) \cap a(j)$ and $\overline{C(i)C(j)}$, and where any other fixed points of x are incident to $\overline{C(i)C(j)}$, any other fixed lines are incident to $a(i) \cap a(j)$. Since $a(i) \cap a(j) = C(D)$ and $\overline{C(i)C(j)} = a(D)$, x fixes $C(D)$ and any other fixed points are on $a(D)$. Similarly, x fixes $a(D)$ and any other fixed lines of x are incident with $C(D)$.

□

Next, some geometry between the involutory centers, d-centers, involutory axes, and d-axes is established.

Lemma 3.12 *$C(i)\ I\ a(D)$ if and only if $i \in D$; $C(D)\ I\ a(i)$ if and only if $i \in D$. There are exactly $\frac{1}{2}(q-1)$ involutory centers incident to $a(D)$ and exactly $\frac{1}{2}(q-1)$ involutory axes incident to $C(D)$.*

Proof: If $C(i)$ I $a(D)$, then i fixes $a(D)$ which implies $i \in D$ by Lemma 3.9. Conversely, if $i \in D$, then since $a(i) \neq a(D)$, $C(i)$ I $a(D)$. Similarly, $C(D)$ I $a(i)$ if and only if $i \in D$.

Since there are precisely $\frac{1}{2}(q-1)$ involutions in D, there are precisely $\frac{1}{2}(q-1)$ involutory centers on $a(D)$ and precisely $\frac{1}{2}(q-1)$ involutory axes through $C(D)$.

□

The points on an involutory axes which are neither involutory centers or d-centers will be shown to be incident to only one involutory axis. This motivates the following definition.

Definition 3.2 *A* singlet *is a point on exactly one involutory axis. A* singline *is a line incident to exactly one involutory center.*

Lemma 3.13 *There are exactly $\frac{1}{2}(q+1)$ d-centers incident to an involutory axis $a(i)$; there are exactly $\frac{1}{2}(q+1)$ d-axes incident to an involutory center $C(i)$. Furthermore, $a(i)$ intersects other involutory axes in either an involutory center or a d-center, and $C(i)$ is joined to other involutory centers by either an involutory axis or a d-axis. The points on $a(i)$ which are neither involutory centers nor d-centers are singlets; the lines through $C(i)$ which are neither involutory axes nor d-axes are singlines.*

Proof: If $i \in D$ then $C(D)$ I $a(i)$ by Lemma 3.12. Using Lemma 3.3, $D \cap \mathcal{C}_{\mathcal{G}}(i) = \langle i \rangle$. $\mathcal{C}_{\mathcal{G}}(i)$ permutes the points of $a(i)$, and $\mathcal{G}_{C(D)} = D$, so the orbit of $C(D)$ on $a(i)$ under $\mathcal{C}_{\mathcal{G}}(i)$ is of length

$$|\mathcal{C}_{\mathcal{G}}(i)|/|\mathcal{C}_{\mathcal{G}}(i) \cap D| = \frac{q+1}{2}.$$

So there are at least $\frac{1}{2}(q+1)$ d-centers on $a(i)$.

Each d-center is incident to $\frac{1}{2}(q-1)$ involutory axes, so each d-center on $a(i)$ is incident to $\frac{1}{2}(q-3)$ involutory axes distinct from $a(i)$. Each involutory center on $a(i)$

is incident to $\frac{q-1}{2}$ involutory axes distinct from $a(i)$. If there were more than $\frac{1}{2}(q+1)$ d-centers on $a(i)$, there would be more than

$$\frac{q+1}{2} \cdot \frac{q-3}{2} + \frac{q+1}{2} \cdot \frac{q-1}{2} + 1 = \frac{1}{2}q(q-1)$$

involutory axes in the plane. From this contradiction, it is seen that there are exactly $\frac{1}{2}(q+1)$ d-centers on $a(i)$.

This count also proves that two involutory axes intersect in either an involutory center or a d-center and that the points on $a(i)$ which are neither involutory centers nor d-centers are singlets.

A similar argument proves the dual statements of the lemma.

□

Lemma 3.13 categorized the points on an involutory axis in terms of involutory centers, d-centers and singlets. Dually it categorized the lines through an involutory center. The next lemma does the same for d-axes and d-centers. There are three possibilities.

Lemma 3.14 *There are* $\frac{q-1}{2}$ *involutory centers on every d-axis* $a(D)$ *and* $\frac{q-1}{2}$ *involutory axes through every d-center* $C(D)$ *corresponding to the* $\frac{q-1}{2}$ *involutions in* D. *In addition, exactly one of the following is true:*

1. *On a d-axis* $a(D)$ *there is exactly one d-center* $C(D)$ *and* $(\frac{q-1}{2})(\frac{q-3}{2})$ *singlets; through a d-center* $C(D)$ *there is exactly one d-axis* $a(D)$ *and* $(\frac{q-1}{2})(\frac{q-3}{2})$ *singlines.*

2. *There are 0 d-centers and* $(\frac{q-1}{2})^2$ *singlets on every d-axis; there are 0 d-axes and* $(\frac{q-1}{2})^2$ *singlines through every d-center.*

3. *There are* $\frac{q-1}{2}$ *d-centers and no singlets on every d-axis; there are* $\frac{q-1}{2}$ *d-axes and no singlines through every d-center.* $C(D)$ *is not incident to* $a(D)$.

Proof: Let $\tilde{D} \cong D_{q-1}$. By Lemma 3.12, there are $\frac{q-1}{2}$ involutory centers incident to $a(\tilde{D})$ and $\frac{q-1}{2}$ involutory axes incident to $C(\tilde{D})$. Each involutory center has $\frac{q+1}{2}$ involutory axes incident to it, and each involutory axis has $\frac{q+1}{2}$ involutory centers incident to it by Lemma 3.13. There are two possibilities: $C(\tilde{D})$ is or is not incident to $a(\tilde{D})$.

Step 1 *Assume* $C(\tilde{D})\ I\ a(\tilde{D})$.

$C(\tilde{D})$ is incident to $\frac{q-1}{2}$ involutory axes, and the $\frac{q-1}{2}$ involutory centers on $a(\tilde{D})$ are each incident to $\frac{q+1}{2}$ involutory axes. So $C(\tilde{D})$ and the involutory centers on $a(\tilde{D})$ account for $(\frac{q+3}{2})(\frac{q-1}{2})$ involutory axes, leaving $(\frac{q-3}{2})(\frac{q-1}{2})$ more involutory axes to intersect $a(\tilde{D})$ from the total of $q(\frac{q-1}{2})$ involutory axes.

Involutory axes intersect in involutory centers or d-centers. There are no more involutory centers incident to $a(\tilde{D})$ but there could conceivably be another d-center on $a(\tilde{D})$. Assume $C(D_1)$ I $a(\tilde{D})$ where $C(D_1) \neq C(\tilde{D})$. $\tilde{D}$ permutes the points of $a(\tilde{D})$ and so the action of $\tilde{D}$ on $C(\tilde{D})$ results in an orbit of $|\tilde{D}|/|\tilde{D}_{C(D_1)}|$ d-centers on $a(\tilde{D})$. $\mathcal{G}_{C(D_1)} = D_1$, and so $\tilde{D}_{C(D_1)} = \tilde{D} \cap D_1$. By Lemma 3.3, $|\tilde{D} \cap D_1| \leq 2$, so the number of points in the orbit of $C(D_1)$ under $\tilde{D}$ is at least $\frac{q-1}{2}$. Each of these centers is incident to $\frac{q-1}{2}$ involutory axes, so there are $\frac{q-1}{2} \cdot \frac{q-1}{2}$ involutory axes incident with this orbit of d-centers. There are only $\frac{q-3}{2} \cdot \frac{q-1}{2}$ involutory axes left to account for, a contradiction which proves that $C(\tilde{D})$ is the only d-center incident to $a(\tilde{D})$.

Involutory axes intersect in involutory centers or d-centers, so the $\frac{q-3}{2} \cdot \frac{q-1}{2}$ remaining involutory axes intersect $a(\tilde{D})$ in $\frac{q-3}{2} \cdot \frac{q-1}{2}$ singlets. Since $a(\tilde{D})$ is incident to $C(\tilde{D})$, a similar argument shows that there will be $\frac{q-3}{2} \cdot \frac{q-1}{2}$ involutory centers which must be joined to $C(\tilde{D})$ in $\frac{q-3}{2} \cdot \frac{q-1}{2}$ distinct singlines and that there can be no d-axes other than $a(\tilde{D})$ incident to $C(\tilde{D})$.

Since the d-groups are conjugate, every d-axis $a(D)$ has exactly one d-center $C(D)$ incident, and conversely, every d-center is incident to exactly one d-axis. Thus, case (1) holds.

Step 2 *Assume for all D that $C(D)$ is not incident to $a(D)$.*

Let $i \in \tilde{D}$. Then $a(i)$ is incident to $C(\tilde{D})$ by Lemma 3.12, and the claim is that $a(i)$ intersects $a(\tilde{D})$ in a singlet or a d-center. Assume by way of contradiction that $a(i)$ intersects $a(\tilde{D})$ in an involutory center $C(j)$. Lemma 3.8 shows $j \in \mathcal{C}_{\mathcal{G}}(i)$, and Lemma 3.12 shows $j \in \tilde{D}$. Since $j \neq i$, this means $\mathcal{C}_{\mathcal{G}}(i) \cap \tilde{D} \geq \langle i, j \rangle$ which contradicts Lemma 3.3. The dual statement is also true: $C(i)$ is joined to $C(\tilde{D})$ by either a singline or a d-axis.

Assume first that $a(i)$ intersects $a(\tilde{D})$ in a d-center. Then all of the $\frac{q-1}{2}$ involutory axes through $C(\tilde{D})$ intersect $a(\tilde{D})$ in d-centers since they are in a line orbit under $\tilde{D}$. So, there is an orbit of $\frac{q-1}{2}$ d-centers on $a(\tilde{D})$, each incident to $\frac{q-1}{2}$ involutory axes. There are also $\frac{q+1}{2}$ involutory axes incident to each of the $\frac{q-1}{2}$ involutory centers on $a(\tilde{D})$, so the involutory centers and d-centers account for the intersection of

$$(\frac{q-1}{2})(\frac{q-1}{2}) + (\frac{q-1}{2})(\frac{q+1}{2}) = q(\frac{q-1}{2})$$

involutory axes with $a(\tilde{D})$. This is the total number of involutory axes, so the $\frac{q-1}{2}$ involutory centers and $\frac{q-1}{2}$ d-centers account for the intersection of all involutory axes with $a(\tilde{D})$. In particular, there are no singlets and no more d-centers on $a(\tilde{D})$.

There are $\frac{q-1}{2}$ d-centers on a d-axis; the next question is how many d-axes are incident to a d-center. Since all d-centers are in one orbit under $\mathcal{G}$, they all have the same number of d-axes. Counting the number of ordered pairs in $J = \{(P, \ell) :$ P is a d-center incident to ℓ a d-axis$\}$ in two ways gives:

(total number of d-centers)(number of d-axes incident to one d-center)

$$=$$

(total number of d-axes)(number of d-centers incident to one d-axis).

Since the total number of d-axes equals the total number of d-centers, there are $(q-1)/2$ d-axes incident to a d-center. This is case (3).

Now it will be shown that case (2) is the only remaining possibility. By assumption, $C(\tilde{D})$ is not incident to $a(\tilde{D})$, and if any involutory axis through $C(\tilde{D})$ intersects $a(\tilde{D})$ in a d-center, then the previous argument shows that case (3) holds. If any involutory center on $a(\tilde{D})$ is joined to $C(\tilde{D})$ by a d-axis then a similar argument also shows that case (3) holds. So the only possibility left is for the involutory axes through $C(\tilde{D})$ to intersect $a(\tilde{D})$ in singlets, and the involutory centers on $a(\tilde{D})$ to be joined to $C(\tilde{D})$ by singlines.

Thus there are $\frac{q-1}{2}$ involutory centers on $a(\tilde{D})$, each with $\frac{q+1}{2}$ involutory axes incident, and there are $\frac{q-1}{2}$ singlets, each with one involutory axis incident. Altogether, this accounts for $(\frac{q+3}{2})(\frac{q-1}{2})$ involutory axes, leaving $(\frac{q-3}{2})(\frac{q-1}{2})$ more involutory axes to intersect $a(\tilde{D})$ from the total of $q(\frac{q-1}{2})$ involutory axes. As in the argument for case (1) these involutory axes cannot intersect $a(\tilde{D})$ in d-centers, so they must intersect $a(\tilde{D})$ in singlets. Thus there are $(\frac{q-1}{2})^2$ singlets on $a(\tilde{D})$ and no d-centers. Similarly, there are $(\frac{q-1}{2})^2$ singlines through $C(\tilde{D})$ and no d-axes.

□

3.2.3 Stabilizer of a Point and a Line

Now enough work has been done to determine the groups which stabilize points. By the partition property for $PSL(2,q)$, a non-trivial element of order prime to p is in either a cyclic group of order $\frac{1}{2}(q+1)$ or in a cyclic group of order $\frac{1}{2}(q-1)$. Using Lemmas 3.7 and 3.11, the fixed points of these elements lie on involutory axes or d-axes. So far nothing has been said about the fixed points of an element of order

p. Result 2.3 lists the subgroups of $PSL(2,q)$; the next lemma establishes which of these subgroups are candidates for point stabilizers.

Lemma 3.15 *Let* P *be a point of the projective plane.*

1. $\mathcal{G}_P \cong D_{q+1}$ *if and only if* P *is an involutory center.*
2. $\mathcal{G}_P \cong D_{q-1}$ *if and only if* P *is a d-center.*
3. $\mathcal{G}_P \cong Z_d$ *where* $1 \neq d \mid \frac{q+1}{2}$ *if and only if* P *is a singlet. In this case* P *is on* $a(i)$ *and* $\mathcal{G}_P \leq \mathcal{C}_\mathcal{G}(i)$ *for an involution* i.
4. *If* P *is not an involutory center, d-center or singlet, then either*
   - $\mathcal{G}_P \cong Z_d$ *where* $1 \neq d \mid \frac{q-1}{2}$, P *lies on a d-axis* $a(D)$ *and* $\mathcal{G}_P \leq D$ *for a d-group* D, *or*
   - $\mathcal{G}_P \cong HL$ *where* H *is elementary abelian of order* p^m, L *is cyclic of order* $t \geq 1$ *where* $t \mid \frac{p^{(m,r)}-1}{2}$, $HL \leq \mathcal{N}_\mathcal{G}(S)$ *for a Sylow p-group* S.

Proof: Since $PSL(2,q)$ is totally irregular on points of the plane, $\mathcal{G}_P$ is non-trivial. If there are at least two involutions in $\mathcal{G}_P$, then P is an involutory center or it lies on at least two involutory axes. By Lemma 3.13, P must be either an involutory center or a d-center. In the first case, $\mathcal{G}_P \cong D_{q+1}$ by Lemma 3.6, and in the second, $\mathcal{G}_P \cong D_{q-1}$ by Lemma 3.9.

If there is only one involution i in $\mathcal{G}_P$, then P is not an involutory center or a d-center. Since i fixes P and $P \neq C(i)$, P is incident to $a(i)$. Since i is the only involution that fixes P, P is a singlet. By Lemma 3.1, the only single-involution subgroups in the list of subgroups given in Result 2.3 are the groups Z_d where $d|\frac{q+1}{2}$.

Now suppose $|\mathcal{G}_P|$ is not divisible by 2. Then Result 2.3 and Lemma 3.1 show that $\mathcal{G}_P$ is isomorphic to one of the following: Z_d where $d|\frac{q+1}{2}$ or $d|\frac{q-1}{2}$, an elementary abelian group of order p^m where $1 \leq m \leq r$, or the semidirect product of an elementary abelian group of order p^m with a cyclic group of order t where $t|\frac{p^{(m,r)}-1}{2}$.

If $\mathcal{G}_P \cong Z_d$ where $d|\frac{q+1}{2}$, then Lemma 3.7 shows that $\mathcal{G}_P \leq \mathcal{C}_{\mathcal{G}}(i)$ for some i and that the fixed points of $\mathcal{G}_P$ are incident to $a(i)$. But then $2 \mid |\mathcal{G}_P|$, a contradiction.

If $\mathcal{G}_P \cong Z_d$ where $d|\frac{q-1}{2}$, then Lemma 3.11 implies that $\mathcal{G}_P \leq D$ for some d-group D and that the fixed points of elements of $\mathcal{G}_P$ lie on $a(D)$.

The last possibility is that $\mathcal{G}_P$ is a semidirect product of an elementary abelian group H of order p^m with a cyclic group L of order t where $t \mid \frac{p^{(m,r)}-1}{2}$, and t could be 1. By Lemma 3.4, since $\mathcal{G}_P \neq PSL(2,p^k)$, $\mathcal{G}_P \leq \mathcal{N}_{\mathcal{G}}(S)$ for some Sylow p-group S.

□

Almost the same result holds for the stabilizer of a line. However, it is possible for the stabilizer of a line to be the identity since there is no restriction that $\mathcal{G}$ acts totally irregularly on lines.

Lemma 3.16 *Let L be a line of the projective plane.*

1. *$\mathcal{G}_L \cong D_{q+1}$ if and only if L is an involutory axis.*

2. *$\mathcal{G}_L \cong D_{q-1}$ if and only if L is a d-axis.*

3. *$\mathcal{G}_L \cong Z_d$ where $1 \neq d \mid \frac{q+1}{2}$ if and only if L is a singline. In this case L is incident to $C(i)$ and $\mathcal{G}_L \leq \mathcal{C}_{\mathcal{G}}(i)$ for an involution i.*

4. *If L is not an involutory axis, d-axis or singline, then either*

   - $\mathcal{G}_L = \langle 1 \rangle$,

   - *$\mathcal{G}_L \cong Z_d$ where $1 \neq d \mid \frac{q-1}{2}$, L is incident to a d-center $C(D)$ and $\mathcal{G}_L \leq D$ for a d-group D, or*

- *$\mathcal{G}_L \cong HM$ where H is elementary abelian of order p^m, M is cyclic of order $t \geq 1$ where $t \mid \frac{p^{(m,r)}-1}{2}$, $HM \leq \mathcal{N}_{\mathcal{G}}(S)$ for a Sylow p-group S.*

3.2.4 Intersections of D-axes

In this section the possibilities for the intersections of d-axes are determined. If there are $\frac{q-1}{2}$ d-centers on a d-axis, then a d-axis $a(D)$ intersects all but $2(q-1)$ d-axes in d-centers or involutory centers. It intersects the remaining $2(q-1)$ d-axes in a point orbit P^D on $a(D)$ where $\mathcal{G}_P$ is divisible by p. Otherwise there are singlets and either 0 or 1 d-centers on a d-axis. In these cases, all but possibly $2(q-1)$ d-axes intersect $a(D)$ in involutory centers or singlets. For the intersection of the remaining $2(q-1)$ d-axes with $a(D)$ there are two possibilities: (1), they intersect $a(D)$ in singlets; (2), they intersect $a(D)$ in a point orbit P^D on $a(D)$ where $\mathcal{G}_P$ is divisible by p.

Lemma 3.17 *Suppose that a d-axis $a(D)$ contains at most one d-center. Then $a(D)$ intersects at least $(\frac{q-3}{2})(\frac{q-1}{2})$ d-axes in singlets, and $C(D)$ is joined to at least $(\frac{q-3}{2})(\frac{q-1}{2})$ d-centers by singlines.*

Proof: By Lemmas 3.14 and 3.12, there are exactly $\frac{1}{4}(q-3)(q-1)$ singlets on $a(D)$ whose incident involutory axes do not contain $C(D)$. If $a(i)$ is such an axis, then i fixes $P := a(i) \cap a(D)$ but moves $a(D)$; thus P lies on a second d-axis.

□

Lemma 3.18 *At most $2(q-1)$ d-axes intersect a d-axis $a(D)$ in points which are not involutory centers, d-centers, or singlets. If $a(D)$ contains more than one d-center, it intersects exactly $2(q-1)$ d-axes in points which are not involutory centers, d-centers or singlets. The dual statements hold.*

Proof: There are $\frac{1}{2}q(q+1)-1$ d-axes other than $a(D)$. By Lemmas 3.12 and 3.13, precisely $\frac{1}{2}(q-1)^2$ d-axes meet $a(D)$ in involutory centers. The desired conclusion follows from Lemmas 3.14 and 3.17 if $a(D)$ contains at most one d-center, and from Lemma 3.14 otherwise.

□

In the next lemma it is shown that if $a(D)$ intersects another d-axis in a point P which is not an involutory center, singlet or d-center, then the order of $\mathcal{G}_P$ is divisible by p and $\mathcal{G}_P$ is determined to some extent. The orbit of P on $a(D)$ under D results in the intersection of $2(q-1)$ d-axes with $a(D)$.

Lemma 3.19 *Let $a(D_1)$ and $a(D_2)$ be distinct d-axes which do not intersect in an involutory center, singlet or d-center. Let $P = a(D_1) \cap a(D_2)$. Either $|\mathcal{G}_P| = 3$ and $p = 3$; or $\mathcal{G}_P = H \cdot L \leq \mathcal{N}_{\mathcal{G}}(S)$ for some Sylow p-group S where $H \leq S$ and L is a cyclic group of order t. For the second possibility, $|H| = p^m$, where $m \geq 1$ and $m|r$; t is odd and equals $\frac{p^m-1}{2}$; $\mathcal{G}_P \cap D_i \cong L$. In both cases, $2(q-1)$ d-axes intersect $a(D_i)$ in points of P^{D_i}.*

Proof: By assumption, P is not incident to an involutory axis and so 2 does not divide $|\mathcal{G}_P|$. Thus by Lemma 3.15 either $\mathcal{G}_P \leq K(D)$ for some d-group D, or $\mathcal{G}_P = H \cdot L$ where H is elementary abelian of order p^m and L is cyclic of order t where $t|\frac{p^{(m,r)}-1}{2}$.

Since P is incident to $a(D_i)$ for $i = 1, 2$, $\mathcal{G}_{a(D_i)} = D_i$ moves P in an orbit of length $|D_i|/|D_i \cap \mathcal{G}_P|$ along $a(D_i)$. $\mathcal{G}_P$ moves $a(D_i)$ in a line orbit through P of length $|\mathcal{G}_P|/|\mathcal{G}_P \cap D_i|$.

Suppose $\mathcal{G}_P \leq K(D)$ for some d-group D and hence fixes $a(D)$. By Lemma 3.11, either $P = C(D)$ or P is incident to $a(D)$. By assumption, $P \neq C(D)$ so P is incident to $a(D)$. If $D_1 \neq D$, then $K(D) \cap D_1 = \langle 1 \rangle$ follows from Lemma 3.3. Then

$|\mathcal{G}_P \cap D_1| = \langle 1 \rangle$ and $\mathcal{G}_P$ moves $a(D_1)$ in a line orbit through P of length $|\mathcal{G}_P|$. So this orbit together with $a(D)$ gives at least $|\mathcal{G}_P| + 1$ d-axes through P. Also since $\mathcal{G}_P$ intersects D_1 trivially, there are $q-1$ points in the point orbit of P under D_1 on $a(D_1)$. Therefore the number of d-axes distinct from $a(D_1)$ accounted for in the orbit of P on $a(D_1)$ equals $(q-1) \cdot |\mathcal{G}_P|$. By Lemma 3.18 there are at most $2(q-1)$ axes left, so $|\mathcal{G}_P| = 2$ which is a contradiction. Thus $D_1 = D$. Using the same argument with D_2 shows $D_1 = D_2$, which is a contradiction. Thus $\mathcal{G}_P \not\leq K(D)$ for any d-group D.

Since $\mathcal{G}_P \not\leq K(D_1)$, $\mathcal{G}_P$ does not fix $a(D_1)$. The number of d-axes that intersect $a(D_1)$ in P^{D_1} (which is contained on $a(D_1)$) is at least

$$\frac{|D_1|}{|D_1 \cap \mathcal{G}_P|} \cdot \left(\frac{|\mathcal{G}_P|}{|\mathcal{G}_P \cap D_1|} - 1\right). \tag{3.1}$$

Suppose $D_1 \cap \mathcal{G}_P = \langle 1 \rangle$. Then formula 3.1 gives $\frac{(q-1)}{1} \cdot (|\mathcal{G}_P| - 1)$ d-axes which intersect $a(D_1)$ in the point orbit P^{D_1}. Since there are at most $2(q-1)$ d-axes which can intersect $a(D_1)$ in this orbit, $|\mathcal{G}_P| = 3$, and exactly $2(q-1)$ d-axes intersect $a(D_1)$ in points of this orbit. Since $\mathcal{G}_P \not\leq K(D)$, $\mathcal{G}_P \leq S$ for some Sylow p-group S which forces $p = 3$. The same result occurs if $D_2 \cap \mathcal{G}_P = \langle 1 \rangle$.

So now assume $\mathcal{G}_P \cap D_i \neq \langle 1 \rangle$ for $i = 1, 2$. Then since $\mathcal{G}_P \not\leq D_i$, $\mathcal{G}_P \cong H \cdot L$ for some $H \leq S$, L a cyclic group of order t where $t | \frac{p^{(m,r)}-1}{2}$, and where $\mathcal{G}_P \leq \mathcal{N}_{\mathcal{G}}(S)$ for some Sylow p-group S. Since $\mathcal{G}_P$ intersects both D_i non-trivially for $i = 1, 2$, $K(D_i) \leq \mathcal{N}_{\mathcal{G}}(S)$, and $\mathcal{G}_P \cap D_i \cong L$.

Putting this information in formula 3.1 gives

$$\frac{(q-1)}{t} \cdot \left(\frac{p^m t}{t} - 1\right) = (q-1)\frac{p^m - 1}{t}$$

d-axes that intersect $a(D_1)$ in the point orbit P^{D_1}. This number is less than or equal to $2(q-1)$, so $\frac{p^m-1}{t} \leq 2$, which implies $t \geq \frac{p^m-1}{2}$. Since $t | \frac{p^m-1}{2}$, $t = \frac{p^m-1}{2}$, and exactly

$2(q-1)$ d-axes intersect $a(D_1)$ in P^{D^1}. Thus, $(m, r) = m$. This implies $m|r$, so m is odd, and t is then also odd.

In both possibilities, the order of $\mathcal{G}_P$ is divisible by p.

□

Definition 3.3 *A* d-singpoint *is a point whose stabilizer is a subgroup of $K(D)$ for some d-group D. A* p-point *is a point whose stabilizer contains an element of order p.*

Lemma 3.20 1. *The set of all points in the plane Π is the disjoint union of the following five sets: the involutory centers, the d-centers, the singlets, the d-singpoints and the p-points.*

2. *A d-singpoint is incident with precisely one d-axis.*

Proof: Conclusion (1) follows from Lemma 3.15. If P is a d-singpoint with stabilizer a subgroup of $K(D)$, then P is incident to $a(D)$ by Lemma 3.11. Since $K(D)$ contains no elements of order p, P is not fixed by an element of order p; so by Lemma 3.19, P is not incident with two distinct d-axes. This proves conclusion (2).

□

Lemma 3.21 1. *If a d-axis $a(D)$ contains a p-point P, the set of all p-points on $a(D)$ is just the point orbit P^D; in this case, $a(D)$ intersects precisely $2(q-1)$ other d-axes in the point of P^D.*

2. *If a d-axis $a(D)$ contains more than one d-center, it contains a p-point.*

Proof: Let $a(D)$ be a d-axis. Suppose $a(D)$ contains a p-point P. Then by Lemma 3.16, $\mathcal{G}_P$ does not fix $a(D)$, and so $a(D)$ intersects another d-axis in P. Since

P is a p-point, Lemma 3.15 shows that P is not an involutory center, d-center, or singlet; thus Lemma 3.19 applies to show that exactly $2(q-1)$ other d-axes intersect $a(D)$ in the points of P^D. By Lemma 3.18, at most $2(q-1)$ d-axes intersect $a(D)$ in points which are not involutory centers, d-centers or singlets, so the set of all p-points on $a(D)$ must be the points in the point orbit P^D. This is conclusion (1).

Now assume that a d-axis $a(D)$ contains more than one d-center. Then Lemma 3.18 applies to show that $a(D)$ intersects $2(q-1)$ other d-axes in points which are not involutory centers, d-centers or singlets, and so, by Lemma 3.19, must be p-points. Thus conclusion (2) holds.

□

Theorem 3.1 Let N_p denote the number of p-points in the plane, and let n_d denote the number of d-singpoints on a d-axis.

1. *Then $n^2+n+1 = q^2 + \frac{1}{2}q(q-1)(n-q) + \frac{1}{2}q(q+1)n_d + N_p$.*
2. *Suppose that the d-axes contain no p-points. Then in case 1 of Lemma 3.14,*

$$n_d = n - \frac{1}{4}(q-1)^2, \text{ and } N_p = n^2 - (q^2-1)n + \frac{1}{8}(q^4+3q^3-13q^2+q+8);$$

in case 2 of Lemma 3.14,

$$n_d = n+1-\frac{1}{4}(q^2-1), \text{ and } N_p = n^2-(q^2-1)n+\frac{1}{8}(q^4+5q^3-17q^2-5q+8);$$

Proof: Lemma 3.20 shall be applied to prove conclusion (1). There are $\frac{1}{2}q(q-1)$ involutory centers and $\frac{1}{2}q(q+1)$ d-centers, hence a total of q^2 points of these two types. By Lemmas 3.8 and 3.13, each of the $\frac{1}{2}q(q+1)$ involutory axes contains precisely $\frac{1}{2}(q+1)$ involutory centers and $\frac{1}{2}(q+1)$ d-centers and, hence, precisely $n-q$ singlets. Thus there are $\frac{1}{2}q(q-1)(n-q)$ singlets in the plane. Since there are

$\frac{1}{2}q(q+1)$ d-axes, there are $\frac{1}{2}q(q+1)n_d$ d-singpoints in the plane. By Lemma 3.20, the remaining points are p-points. Thus conclusion (1) holds.

Suppose now that the d-axes contain no p-points. Then, by Lemma 3.21, there is at most one d-center on a d-axis, and cases 1 and 2 of Lemma 3.14 hold. For case (1), each dqmin axis is incident with exactly one d-center, $\frac{1}{2}(q-1)$ involutory centers, and $\frac{1}{4}(q-1)(q-3)$ singlets. This implies there are exactly $n_d = n - \frac{1}{4}(q-1)^2$ d-singpoints on $a(D)$. In case (2) of Lemma 3.14, each d-axis is incident with 0 d-centers, $\frac{1}{2}(q-1)$ involutory centers, and $\frac{1}{4}(q-1)^2$ singlets. Thus, in this case $n_d = n + 1 - \frac{1}{4}(q^2-1)$. Substitution of these values of n_d into the equation of conclusion (1) yield the formulas for N_p given in conclusion (2).

□

3.3 Theorems

The geometric structure induced by the existence of an involutory homology together with the totally irregular action of $PSL(2,q)$ on the plane will now be used to deduce some results. A bound for the order of the plane is found. It is shown that $PSL(2,q)$ acts strongly irreducibly or leaves invariant a Desarguesian subplane of order q. The order of the plane is found in the case where every element of order p acts fixed-point-freely.

3.3.1 Bound for the Order of the Plane

At this point there is enough information to get a rough bound on the order of the plane. The points in the plane are partitioned into five sets by Lemma 3.20. The number of involutory centers, d-centers, and singlets are known. The number of d-singpoints and p-points are variable, but an upper bound is found. The next lemma restricts the possibilities for $\mathcal{F}(x)$, where x is an element of order p.

Lemma 3.22 *An element x of order p is not a perspectivity nor is $\mathcal{F}(x)$ a Baer subplane.*

Proof: Let x be an element of order p. If x is a perspectivity or if $\mathcal{F}(x)$ is a Baer subplane, then every line of Π is incident to one of the fixed points of x. In particular an involutory axis $a(i)$ is incident to a point P of $\mathcal{P}(x)$, and so P is fixed by the involution i. Thus $\langle i, x\rangle \leq \mathcal{G}_P$, but by Lemma 3.15, $2p \not| \, |\mathcal{G}_P|$.

Theorem 3.2 *The order $n < \frac{p}{p-1}(q^2-1) \leq \frac{3}{2}(q^2-1)$.*

Proof: By Lemma 3.22 and Theorem 2.3, a planar group element of order p fixes the m^2+m+1 points of a non-Baer subplane, and $m^2+m+1 \leq n+1$. A non-planar element of order p is a generalized perspectivity, but not a perspectivity, and thus also fixes at most $n+1$ points. There are exactly $(q-1)/(p-1)$ subgroups of order p in each of the $q+1$ Sylow p-subgroups, so $N_p \leq \frac{1}{(p-1)}(q^2-1)(n+1)$. Since each d-axis is incident with exactly $\frac{1}{2}(q-1)$ involutory centers, $n_d \leq n+1-\frac{1}{2}(q-1)$. Using these bounds for N_p and n_d, Theorem 3.1 yields the inequality

$$n^2+n+1 \leq q^2+\frac{1}{2}q(q-1)(n-q)+\frac{1}{2}q(q+1)(n+1)-\frac{1}{4}q(q^2-1)+\frac{1}{(p-1)}(q^2-1)(n+1).$$

So

$$\begin{aligned} n^2+(1-q^2-\frac{q^2-1}{p-1})n &\leq -1+q^2-q^2\left(\frac{q-1}{2}\right)-\frac{q(q+1)(q-3)}{4}+\frac{q^2-1}{p-1} \\ &< 0. \end{aligned}$$

This then gives that

$$n < (q^2-1)+\frac{1}{p-1}(q^2-1) = \frac{p}{p-1}(q^2-1) \leq \frac{3}{2}(q^2-1) \text{ as } p \geq 3.$$

□

This is certainly a rough bound, since there is overlap between the points on the involutory axes and d-axes. The merit in this bound is in its ease of expression.

3.3.2 $PSL(2,q)$ Acts Strongly Irreducibly or Fixes a Subplane of Order q

A motivating factor for the study of $PSL(2,q)$ on projective planes was the work of Hering and others on strongly irreducible action. This theorem shows that if $PSL(2,q)$ acts totally irregularly and contains an involutory homology, then $PSL(2,q)$ acts strongly irreducibly or leaves invariant a Desarguesian subplane of order q.

Theorem 3.3 *Assume $PSL(2,q)$ $q \equiv 3 \pmod 4$, acts totally irregularly and contains an involutory homology. Then either $PSL(2,q)$ acts strongly irreducibly on Π, or $PSL(2,q)$ leaves invariant a Desarguesian subplane of order q.*

Proof: By Theorem 2.4, $PSL(2,q)$ fixes no point or line. If $PSL(2,q)$ leaves a triangle invariant, then since $PSL(2,q)$ is simple, the kernel of the action is $PSL(2,q)$, a contradiction. Thus $PSL(2,q)$ leaves no triangle invariant.

Suppose $PSL(2,q)$ fixes a subplane Π_o of order m. Then by Theorem 2.1, the centers and axes of perspectivities lie in Π_o. So the involutory centers and axes are in Π_o. Then, by Lemma 3.13, the d-centers and d-axes must also be in the subplane. Since there are $\frac{q+1}{2}$ involutory centers and $\frac{q+1}{2}$ d-centers on an involutory axis, $m \geq q$. If $m = q$, then Theorem 2.2 gives that the subplane is Desarguesian.

So assume $m \neq q$. By Lemma 3.14, if a d-axis has at most one d-center, then there are at least $(\frac{q-1}{2})^2 + 1$ points on a d-axis $a(D)$, each point either a singlet, a d-center or an involutory center. Since these points are the intersections of $a(D)$ with involutory axes, and hence in Π_o, $m \geq (\frac{q-1}{2})^2$. Then Theorem 2.3 shows $n \geq (\frac{q-1}{2})^4$. By Theorem 3.2, $n < \frac{3}{2}q^2$; this contradiction proves there is more than one d-center on a d-axis.

By Lemma 3.14 there are then $\frac{q-1}{2}$ d-centers on a d-axis. By Lemmas 3.8, 3.13 and 3.14, there are $\frac{q+1}{2}$ involutory centers on an involutory axis $a(i)$, each with $\frac{q+1}{2}$ d-axes incident, and $\frac{q+1}{2}$ d-centers on an involutory axis, each with $\frac{q-1}{2}$ d-axes incident. These points account for all of the intersections of d-axes with $a(i)$. By Lemma 3.13, involutory axes meet in either an involutory center or a d-center, so the involutory centers and d-centers also account for all of the intersections of involutory axes with $a(i)$. Altogether, there are $q+1$ involutory centers and d-centers incident to $a(i)$, so since $m > q$, there exists a point P in Π_o and on $a(i)$ which is not an involutory center or a d-center, and hence is not incident to any d-axis or any involutory axis other than $a(i)$. Since by Lemma 3.13 involutory centers are joined by d-axes and involutory axes, this means that P must be joined to the $(\frac{q(q-1)}{2} - \frac{q+1}{2})$ involutory centers not on $a(i)$ by distinct lines of Π_o. So now counting the number of lines of Π_o through P gives that

$$m+1 \geq (\frac{q(q-1)}{2} - \frac{q+1}{2} + 1) \text{ and so } m+1 \geq \frac{q^2-2q+1}{2}.$$

Then by Theorem 2.3, $n \geq (\frac{(q^2-2q-1)^2}{4})$, which contradicts the upper bound of $\frac{3}{2}q^2$ for n.

□

3.3.3 Assume No Point is Fixed by an Element of Order p

So far it has been shown that if a non-trivial element of $\mathcal{G}$ has order prime to p then its fixed points are involutory centers, d-centers, or lie on involutory axes or d-axes. The next theorem looks at the cases where the elements of order p fix no point.

Theorem 3.4 *Assume the stabilizer of every point P is prime to p. Then either $C(D)$ is incident to $a(D)$, and*

$$n = (\frac{q^2-1}{2}) \pm \sqrt{\frac{1}{8}(q^4 - 3q^3 + 9q^2 - q - 6)};$$

or $C(D)$ is not incident to $a(D)$ and

$$n = (\frac{q^2-1}{2}) \pm \sqrt{\frac{1}{8}(q^4 - 5q^3 + 13q^2 + 5q - 6)}.$$

If, in addition, the stabilizer of every point is even, then $q = 7$ and Π is the Desarguesian plane of order 9 or 11. The Desarguesian planes of order 9 and 11 do admit $PSL(2,7)$ as a totally irregular collineation group with an involutory homology.

Proof: By assumption, $N_p = 0$. Thus, by Lemma 3.21, a d-axis contains at most one d-center. The formulas for n then follow from conclusion (2) of Theorem 3.1.

Now assume also that the stabilizer of every point has even order. Then every point is fixed by some involution, and so is either an involutory center or incident to an involutory axis. Thus, by Lemma 3.13, the points of Π are involutory centers, d-centers, or singlets; there are no d-singpoints and $n_d = 0$. Substituting 0 for n_d in the formulas of conclusion (2) of Theorem 3.1, and using the fact that 2 is a unit modulo q, yields the conclusion that n is congruent to $\frac{1}{4}$ or $\frac{-5}{4}$ modulo q. Part (1) of Theorem 3.1 shows that $n^2 + n + 1 \equiv 0 \pmod{q}$; substitution gives that $\frac{21}{16} \equiv 0 \pmod{q}$ in both cases. For 21 to be divisible by a prime power $q > 3$, q must be 7. Plugging this into the two formulas for n_d in conclusion (2) of Theorem 3.1 yields the results of $n = 9$ and $n = 11$. Chat Yin Ho has proved in [8] and [7], respectively, that the planes of orders 9 and 11 which admit $PSL(2,7)$ are Desarguesian.

Now the question is whether the Desarguesian planes of order 9 and 11 admit $PSL(2,7)$ as a totally irregular collineation group with involutory homology. If $PSL(2,7)$ is a subgroup of the collineation group of a Desarguesian plane of order

q, then $PSL(2,7)$ is a subgroup of $PSL(3,q)$. For q equal to 9 and 11, it is known that $PSL(2,7)$ is a subgroup of $PSL(3,q)$ [1]. Thus the elements of $PSL(2,7)$ can be represented by matrices, and so no involution in $PSL(2,7)$ is planar. Since 9 and 11 are odd, and involutions are either planar or are perspectivities, an involution in $PSL(2,7)$ must be an involutory homology. There can be no semi-regular orbit of $PSL(2,7)$ on the points of Π since the order of $PSL(2,7)$ is 168, and the number of points in the planes of order 9 and 11 are 91 and 133, respectively.

□

For a particular q, there are only four possibilities for n. The radicand must be a square. For $q = 7$, this is true and n is 9, 39, 11 or 37. When $q = 11$, the radicand is not a square. One observation that cuts down on the possibilities for values of q that make the radicand a square is achieved by looking at the radicand mod q.

Corollary 3.1 *If the stabilizer of every point is p-prime then* -3 *is a square mod* q.

CHAPTER 4
CONTRIBUTION OF ELEMENTS OF ORDER p

The geometric structure due to the involutory centers and axes has been determined to some extent. If no point is fixed by an element of order p, Theorem 3.4 showed four possibilities for n in terms of q. Now consider the case where an element of order p fixes some point. First it will be established that the elements of order p have equivalent fixed point/line structures. The element of order p is conceivably a generalized homology or elation, or else it is triangular or planar. Each of these cases is studied. The element of order p is proved to not be triangular, nor to be a generalized homology. If it is a generalized elation then Π is a Desarguesian plane of order q; $n = q^2$, and there is a Desarguesian subplane of order q left invariant by $PSL(2,q)$, and Π is non-Desarguesian; or $p = 3$. The planar case is ruled out when $\mathcal{N}_\mathcal{G}(S)$ fixes a line for a Sylow p-group S. It is also ruled out when $\mathcal{N}_\mathcal{G}(S)$ fixes a point and q is prime. $\mathcal{N}_\mathcal{G}(S)$ is shown to fix at most one point and one line, so the cases that remain are when $\mathcal{N}_\mathcal{G}(S)$ fixes no point and no line, and when $\mathcal{N}_\mathcal{G}(S)$ fixes a point and q is not prime. Partial results were found for both of these cases, but will not be put in this thesis.

The first section establishes some basic structural results. The next section gives restrictions on a triangular p-group; in particular, it is shown that no element of order p is triangular. The non-triangular cases which have been solved rely on the assumption that $\mathcal{N}_\mathcal{G}(S)$ fixes either a point or a line, so one section is on the geometry that results from this assumption. The next section deals with the case where the element of order p is a generalized perspectivity, and the final section is about the case where it is planar.

4.1 Base Lemmas

First it is established that all elements of order p have equivalent fixed point/line substructures.

Lemma 4.1 *The subgroups of order p are in one conjugacy class. The elements of order p have equivalent fixed point/line substructures. If H is a subgroup of order p contained in a Sylow p-group S, then* $\mathcal{N}_{\mathcal{G}}(H)$ *is a semidirect product of S with a cyclic group L of order* $\frac{p-1}{2}$. *For each* $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$ *there exists a* $g \in S$ *such that* $L^g = \mathcal{N}_{\mathcal{G}}(H) \cap K(D)$. *For each* $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$, *there is a subgroup L in K(D) of order* $\frac{p-1}{2}$ *which normalizes each subgroup of order p in S. If* $p \neq 3$, *L is non-trivial.*

Proof: Let S be a Sylow p-group. By Result 2.2, the elements of order p in S separate into two conjugacy classes, and since r is odd by Lemma 3.1, the non-trivial elements of a cyclic group of order p are separated into the two conjugacy classes. Thus a subgroup of order p has representatives from each conjugacy class, and so the subgroups of order p form one conjugacy class.

The non-trivial elements of the subgroup of order p have the same fixed points and lines since the order is prime. Thus the fixed point/line structure of an element x of order p is the common fixed point/line structure for all of the elements in $\langle x \rangle$, and since subgroups of order p are conjugate, the subgroups have isomorphic fixed point/line structures. Thus the elements of order p have isomorphic fixed point/line structures.

$\mathcal{N}_{\mathcal{G}}(S)$ is a Frobenius group with kernel S and complement $K(D_1)$ for some d-group D_1 (see, for instance, page 41 of Gorenstein [4]); there are q distinct conjugates of $K(D_1)$ under S which are necessarily in $\mathcal{N}_{\mathcal{G}}(S)$. By Result 2.2, if $|H| = p$ and

$H \leq S$, then $\mathcal{N}_{\mathcal{G}}(H)$ is a semidirect product of S with a cyclic group L of order $\frac{p-1}{2}$ where $L \leq K(D_1)$. The action of S on L results in q distinct conjugates of L contained in $\mathcal{N}_{\mathcal{G}}(H)$; so if $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$, there exists a $g \in S$ such that $L^g \leq K(D)$ and $\mathcal{N}_{\mathcal{G}}(H) \cap K(D) = L^g$.

The last paragraph showed that there are q cyclic subgroups, one from each $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$, of order $\frac{p-1}{2}$ in $\mathcal{N}_{\mathcal{G}}(H)$ for each H a subgroup of order p. There are only q subgroups of order $\frac{p-1}{2}$ in $\mathcal{N}_{\mathcal{G}}(S)$, so each one must be in the normalizer of each subgroup of order p.

□

The elements of order p have equivalent fixed point/line substructures; they might share the same one, or the structures might be distinct or intersect. The next lemma establishes what connection the fixed points and lines of an element of order p have with d-axes and d-centers, and shows that no point fixed by an element of order p is on an involutory axis, and that no line fixed by an element of order p is incident with an involutory center,

Lemma 4.2 *Let x be an element of order p. No point fixed by x is incident to an involutory axis; no line fixed by x is incident with an involutory center. Let $\langle 1 \rangle \neq M \leq K(D)$ and $\langle 1 \rangle \neq H \leq S$ for S a Sylow p-group. Then M and H have at most one fixed point P and one fixed line L in common. The point, if it exists, will be incident to $a(D)$, with its stabilizer a subgroup of $\mathcal{N}_{\mathcal{G}}(S)$; the line, if it exists, will be incident with $C(D)$ and with its stabilizer in $\mathcal{N}_{\mathcal{G}}(S)$. If $\langle M, H \rangle \leq \mathcal{G}_P$, then no non-trivial subgroup of $K(D)$ fixes a point of $\mathcal{F}(H)$ distinct from P; if $\langle M, H \rangle \leq \mathcal{G}_L$, then no non-trivial subgroup of $K(D)$ fixes a line of $\mathcal{F}(H)$ distinct from L. The points (and lines) fixed by S fall into orbits of length $\frac{q-1}{2}$, with the possible exception of one* *orbit of length one. The number of fixed points (lines) of $\mathcal{F}(S)$ is congruent to 0 or 1* $\pmod{\frac{q-1}{2}}$.

Proof: Let x, M, $K(D)$, S, H be as stated above. If P is fixed by x and P is incident to an involutory axis, then $\mathcal{G}_P$ is divisible by $2p$ contradicting Lemma 3.15. Similarly, a line fixed by x cannot be incident with an involutory center by Lemma 3.16.

Now suppose P is a point fixed by both M and H. By Lemma 3.11, if P is a point fixed by M, then $P = C(D)$ or P lies on $a(D)$. By Lemma 3.9, $\mathcal{G}_{C(D)} = D$, so H cannot fix $C(D)$. So P lies on $a(D)$. By Lemma 3.15, $\mathcal{G}_P \leq \mathcal{N}_{\mathcal{G}}(S)$.

If there exists a point P such that $\mathcal{G}_P \geq \langle M, H \rangle$, then if another point Q of $\mathcal{F}(H)$ were fixed by a non-trivial subgroup of $K(D)$, Q would also be incident to $a(D)$. Thus H would fix $a(D)$ and so $H \leq \mathcal{G}_{a(D)} = D$. This is not true, and so in particular, there is at most one fixed point P in common for H and M.

A similar argument works for lines using 3.16.

Since $\mathcal{N}_{\mathcal{G}}(S)$ is the semidirect product of $K(D)$ with S for some d-group D, $\mathcal{N}_{\mathcal{G}}(S)$ can fix at most one point and one line. $K(D)$ permutes the fixed points (lines) of S. If a non-trivial subgroup M_1 of $K(D)$ fixes a point P of $\mathcal{F}(S)$, then it fixes exactly one; since $K(D)$ normalizes M_1, $K(D)$ fixes P also. P is the only point in $\mathcal{F}(S)$ is fixed by $K(D)$, so under $K(D)$ there is one orbit of length 1 and the rest are of length $\frac{q-1}{2}$. Otherwise, if no non-trivial subgroup of $K(D)$ fixes a point of $\mathcal{F}(S)$, the orbits of points under $K(D)$ are of length $\frac{q-1}{2}$.

□

Lemma 4.3 *Let x be an element of order p and S its Sylow p-group. Let $\mathcal{C} = (\mathcal{P}(x), \mathcal{L}(x))$ be the configuration of fixed points and lines of x. Assume $\mathcal{C} \neq \emptyset$ and let H be the kernel of the action of S on $\mathcal{C}$. If $|H| = p^t$, then $t|r$, $\mathcal{N}_{\mathcal{G}}(H) = S \cdot T$,* *and T is cyclic of order $\frac{|H|-1}{2}$. $|\mathcal{N}_{\mathcal{G}}(H)| = q\frac{|H|-1}{2}$. Distinct conjugates of H intersect in the identity.*

Proof: Let x, S, C, and H be as above. Since the fixed point/line configuration of x is $\mathcal{C}$, then by Lemma 4.1, every element of order p has a configuration of fixed points and lines which is equivalent to $\mathcal{C}$; in particular each element of order p has the same number of fixed points. If $1 \neq y \in H \cap H^g$ for $H^g \leq S$, then y fixes the points of $\mathcal{C}$ and the points of $\mathcal{C}^g$. Since the number of fixed points of y equals the number of points of $\mathcal{C}$, $\mathcal{C} = \mathcal{C}^g$. If $\mathcal{C} = \mathcal{C}^g$ then H^g is contained in the kernel of the action of S on $\mathcal{C}$, i.e. $H^g \leq H$, which forces $H = H^g$ since the two subgroups have the same order. So a conjugate H^g intersects H in the identity or is H. In general, distinct conjugates of H intersect in the identity, since $|H^k \cap H^f| = |H^{kf^{-1}} \cap H|$.

Let $1 \neq y \in S$. Lemma 4.1 shows that all subgroups of order p are conjugate, and so there is some element $g \in \mathcal{N}_{\mathcal{G}}(S)$ where $\langle x \rangle^g = \langle y \rangle$. Thus $H^g \geq \langle x \rangle^g = \langle y \rangle$. So every element in S is in one of the conjugates of H, and no two distinct conjugates share an element of order p. There are $|H| - 1$ elements of order p in a conjugate of H and $q - 1$ elements of order p in S, so there must be $\frac{q-1}{|H|-1}$ distinct conjugates of H in S.

Now count the number of distinct conjugates of H in S using $\mathcal{N}_{\mathcal{G}}(H)$. A conjugate $H^g \leq S \leftrightarrow g \in \mathcal{N}_{\mathcal{G}}(S)$ by Lemma 3.4, and $H^g = H \leftrightarrow g \in \mathcal{N}_{\mathcal{G}}(H)$, so the number of distinct conjugates of H in S is

$$\frac{|\mathcal{N}_{\mathcal{G}}(S)|}{|\mathcal{N}_{\mathcal{G}}(H)|} = \frac{q\frac{q-1}{2}}{|\mathcal{N}_{\mathcal{G}}(H)|}.$$

Putting these two ways to count the conjugates of H together gives that $|\mathcal{N}_{\mathcal{G}}(H)| = q\frac{|H|-1}{2}$. Since $\mathcal{N}_{\mathcal{G}}(H) \leq \mathcal{N}_{\mathcal{G}}(S)$, $(|H| - 1)$ divides $(q - 1)$ which means $t|r$.

Result 2.2 states that $\mathcal{N}_{\mathcal{G}}(H)$ is a semidirect product of S with a cyclic group T, so the rest of the lemma follows.

Lemma 4.4 *Let x be an element of order p in a Sylow p-group S. If x is a generalized perspectivity and H is maximal in S with respect to having $C(x)$ as a common center (or $a(x)$ as a common axis) then $|H| = p^t$, $t|r$ $\mathcal{N}_{\mathcal{G}}(H) = S \cdot T$, where T is cyclic, and $|\mathcal{N}_{\mathcal{G}}(H)| = q^{\frac{|H|-1}{2}}$. Every non-trivial element of S is an element of exactly one of the distinct conjugates of H in S.*

Proof: The proof is similar to Lemma 4.3. Distinct conjugates of H intersect trivially, since otherwise they would share the same generalized center contradicting the maximality of H. Every element of order p in S must be in one conjugate of H, and so the distinct conjugates of H therefore partition the $q-1$ elements of order p in S. Counting as in the previous lemma gives the result.

□

4.2 Triangular p-subgroups

First it will be established that if a p-group fixes a triangle pointwise then $p = 3$. Then it will be shown that no element of order p is triangular.

Lemma 4.5 *If a p-subgroup H of a Sylow p-group S fixes a triangle $\mathcal{T}$ pointwise then $p = 3$ and the kernel of the action of S on the triangle is a group M where* $|S|/|M| = 3$.

Proof: Assume H is triangular, fixing the triangle $\mathcal{T}$ pointwise, and let S be the Sylow p-group that contains H. Let M be the kernel of the action of S on $\mathcal{T}$.

Step 1 *Assume $M = S$.*

Let $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$. By Lemma 4.2, the number of fixed points of S is congruent to 1 or 0 mod$\frac{q-1}{2}$. So $\frac{q-1}{2} = 2$ or $\frac{q-1}{2} = 3$ which implies $q = 5$ or $q = 7$. Since $q \equiv 3 \pmod 4$, $q = 7$, and the stabilizer of each point of the triangle $\mathcal{F}(S)$ is S. The equation from conclusion (1) of Theorem 3.1 will now be used to come up with a contradiction. Since there are 8 Sylow p-subgroups, each fixing a distinct triangle of points, $N_P = 24$. Next, possible values of n_d are found.

Suppose a p-point P is incident to a d-axis $a(D)$. Since $\mathcal{G}_P \cap D = \langle 1 \rangle$, $\mathcal{G}_P$ moves $a(D)$ in an orbit of 7 d-axes through P. In particular, P is the intersection of two d-axes in a point which is not an involutory center, d-center, or a singlet. However, $\mathcal{G}_P$ does not satisfy the conclusions of Lemma 3.19, a contradiction which proves there is no p-point incident to a d-axis. Then by Lemma 3.21, there is at most one d-center on a d-axis, and Theorem 3.1 gives equations for N_P in conclusion (2). Substituting 24 for N_P and 7 for q yields non-integer solutions for n. This contradiction shows $q \neq 7$; and $M \neq S$.

Step 2 *Assume $M < S$.*

$S \leq \mathcal{N}_\mathcal{G}(M)$ and so S permutes the fixed points of M. Let $x \in S \backslash M$. If x fixes one point of $\mathcal{F}(M)$, then it must move the other two in an orbit of length two which means $p = 2$, contrary to the assumption that $p \equiv 3 \pmod 4$. So x moves the three points of the triangle in an orbit of length three. Thus, the stabilizer in S of each point in the triangle is precisely M, and S moves the three points in $\mathcal{F}(M)$ in orbit(s) of length $|S|/|M|$. This implies $|S|/|M| = 3$ which in turn implies $p = 3$.

□

Theorem 4.1 No element of order p is triangular.

Proof: Assume an element x of order p is triangular. Then $\langle x \rangle$ is triangular and so by Lemma 4.5, $p = 3$, and M is the kernel of the action of S on the triangle where $|S|/|M| = 3$. Since $q > 3$ and q is an odd power of 3, $|S| \geq 27$, and hence $M \cap M^g \neq \langle 1 \rangle$. Then Lemma 4.3 applies to show that there is only one distinct conjugate of M in S, and so $\mathcal{N}_\mathcal{G}(M) = \mathcal{N}_\mathcal{G}(S)$. However, Lemma 4.3 also shows that

$|\mathcal{N}_{\mathcal{G}}(M)| = q^{\frac{|M|-1}{2}}$, a contradiction which proves that the element of order 3 is not triangular.

□

4.3 $\mathcal{N}_{\mathcal{G}}(S)$ Fixes a Line or a Point

If the element of order p is a generalized perspectivity, it will be shown that $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line or a point. Lemma 4.2 showed that at most one point and line are fixed by $\mathcal{N}_{\mathcal{G}}(S)$. The planar case will be solved in the case that a line is fixed by $\mathcal{N}_{\mathcal{G}}(S)$, and there are some results when a point is fixed by $\mathcal{N}_{\mathcal{G}}(S)$. This section develops the geometry induced when a line or point is fixed by $\mathcal{N}_{\mathcal{G}}(S)$.

Lemma 4.6 *If $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line $\mathcal{L}$ (respectively, a point P) for some Sylow p-group S, then $\mathcal{G}_{\mathcal{L}} = \mathcal{N}_{\mathcal{G}}(S)$ ($\mathcal{G}_P = \mathcal{N}_{\mathcal{G}}(S)$), and for each Sylow p-group S, $\mathcal{N}_{\mathcal{G}}(S)$ fixes exactly one line (respectively, point). Furthermore, the normalizers of distinct Sylow p-groups fix distinct lines (respectively, points), and the lines (points) are distinct from the involutory axes and d-axes (respectively, centers).*

Proof: Let S_1 be a Sylow p-group, and assume $\mathcal{N}_{\mathcal{G}}(S_1)$ fixes a line $\mathcal{L}$. By Lemma 4.2, $\mathcal{N}_{\mathcal{G}}(S_1)$ fixes at most one line and so exactly one line. Since the normalizers of Sylow groups are conjugate, each normalizer of a Sylow-p group S must fix a line $\mathcal{L}_S$. These lines are distinct since if $\mathcal{L}_{S_1} = \mathcal{L}_S$ for $S_1 \neq S$, then $\mathcal{G}_{\mathcal{L}} \geq \langle S_1, S \rangle = \mathcal{G}$, contradicting Lemma 3.16. If $\mathcal{L}_{S_1}$ were an involutory axis or d-axis it would be fixed by an involution i, and $\mathcal{G}_{\mathcal{L}} \geq \langle i, S_1 \rangle$ which again contradicts Lemma 3.16.

The same type of argument, using Lemma 3.15 instead of Lemma 3.16, works for points.

□

Definition 4.1 *A line (point) fixed by* $\mathcal{N}_{\mathcal{G}}(S)$ *is called* L_S (P_S), *and it will be referred to as a* s-axis (s-center).

Next is to determine how L_S and P_S interact with the involutory centers and axes and the d-centers and axes. Although the involutory centers and axes, and d-centers and d-axes exist, the s-axis and the s-center may or may not exist. It is also possible for the s-axis to exist but not the s-center, and vice versa.

Lemma 4.7 *If* L_S *exists, then* $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$ *if and only if* $C(D)$ *is incident to* L_S; *if* P_S *exists then* $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$ *if and only if* $a(D)$ *is incident with* P_S. *Given the existence of the s-axes, there are* q *d-centers incident to a s-axis* L_S *corresponding to the* q *d-groups which normalize* S, *and there are two s-axes through a d-center* $C(D)$ *which correspond to the Sylow* p*-groups which* $K(D)$ *normalizes. Given the existence of the s-centers, there are two s-centers on a d-axis* $a(D)$ *corresponding to the two Sylow* p*-groups that* $K(D)$ *normalizes, and* q *d-axes incident to a s-center* P_S *which correspond to the* q *d-groups that normalize* S. *A s-axis intersects another in a d-center; a s-center is joined to another by a d-axis. An involutory axis intersects a s-axis in a d-center; an involutory center is joined to a s-center by a d-axis.*

Proof: Assume that there is a line L_S fixed by $\mathcal{N}_{\mathcal{G}}(S)$. If $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$ then L_S is fixed by $K(D)$. By Lemma 3.11, $K(D)$ fixes the line $a(D)$, and the other lines fixed by $K(D)$ are incident to $C(D)$. Since $L_S \neq a(D)$ by Lemma 4.6, L_S is incident to $C(D)$. There are q $K(D)$'s in $\mathcal{N}_{\mathcal{G}}(S)$, so there are q d-centers incident to L_S. By Result 2.2 $K(D)$ normalizes exactly two Sylow p-groups, so each d-center is incident to two s-axes. Thus each d-center on L_S is incident to at least one other s-axis other than L_S itself, accounting for q s-axes in the intersection of s-axes with L_S. There are only $q+1$ s-axes, so the d-centers are precisely the points of intersection of L_S

with the other s-axes. So two s-axes intersect in a d-center, and there are exactly two s-axes incident to each d-center. This also shows that there are at most q d-centers on a s-axis, and that if $C(D)$ is incident to L_S, then $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$.

Now look at the intersection of involutory axes with L_S. There are $\frac{q-1}{2}$ involutory axes incident to each d-center by Lemma 3.12, and so the q d-centers on L_S account for $\frac{q(q-1)}{2}$ involutory axes. Since this is the total number of involutory axes, an involutory axis intersects a s-axis in a d-center.

If $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point P_S, the dual results are proved in the same way.

□

In the next four lemmas, the possible intersections of L_S with d-axes are determined, and dually, the type of lines which join P_S to a d-center. If $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$, $a(D)$ intersects L_S in a point fixed by $K(D)$ where the point can be either P_S, $C(D)$ or some other point.

The case where $K(D) \not\leq \mathcal{N}_{\mathcal{G}}(S)$ forces more geometric structure through the intersection of L_S and $a(D)$. Either there are $\frac{q-1}{2}$ d-centers on $a(D)$, and L_S intersects $a(D)$ in one of them, or $p = 3$ and L_S intersects $a(D)$ in a point whose stabilizer is of order three. The dual results are proved with P_S. In particular the existence of P_S or L_S forces the situation of $\frac{q-1}{2}$ d-centers on a d-axis and $\frac{q-1}{2}$ d-axes through a d-center when $p \neq 3$.

Lemma 4.8 *Suppose* $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$. *If* $\mathcal{N}_{\mathcal{G}}(S)$ *fixes a line* L_S *and* $L_S \cap a(D) = P$, *then either* $\mathcal{G}_P = D$ *and* $P = C(D)$, *or* $\mathcal{G}_P = \mathcal{N}_{\mathcal{G}}(S)$ *and* $P = P_S$, *or* $\mathcal{G}_P = K(D)$. *If* $\mathcal{N}_{\mathcal{G}}(S)$ *fixes a point* P_S *and* $\overline{P_S C(D)} = L$, *then either* $\mathcal{G}_L = D$ *and* $L = a(D)$, *or* $\mathcal{G}_L = \mathcal{N}_{\mathcal{G}}(S)$ *and* $L = L_S$, *or* $\mathcal{G}_L = K(D)$.

Proof: If $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$ then L_S is fixed by $K(D)$. $K(D)$ also fixes $a(D)$ so L_S intersects $a(D)$ in a point P whose stabilizer contains $K(D)$. By Lemma 3.15, either

$\mathcal{G}_P = D$ and $P = C(D)$, $\mathcal{G}_P = \mathcal{N}_{\mathcal{G}}(S)$, or $\mathcal{G}_P = K(D)$. By Lemma 4.6, if P is fixed by $\mathcal{N}_{\mathcal{G}}(S)$, then that point is unique and is P_S.

A similar argument when $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point P_S, using Lemma 3.16, gives the rest of the results.

□

The following lemma is very important in establishing an upper bound to the number of points on a line.

Lemma 4.9 *There are q d-centers on L_S, and possibly q more points where $a(D)$ intersects L_S for $K(D)$ in $\mathcal{N}_{\mathcal{G}}(S)$. The remaining points on L_S have as stabilizers non-trivial subgroups of S.*

Proof: Lemma 4.7 showed that there are q d-centers on a s-axis L_S, and it was shown that the q other s-axes will intersect L_S in these points. If $a(D)$ is not incident to $C(D)$, Lemma 4.8 shows that $a(D)$ will intersect L_S in a point whose stabilizer contains $K(D)$.

Let P be a point on L_S that is not a d-center nor is incident to $a(D)$ for $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$. Since $\mathcal{G}$ is totally irregular on points, P has a non-trivial stabilizer. Thus there are $|\mathcal{G}_P|/|\mathcal{G}_P \cap \mathcal{G}_{L_S}|$ lines in the line orbit of L_S under $\mathcal{G}_P$, the lines incident with P. Since s-axes intersect in d-centers, and P is not a d-center, there is exactly one s-axis through P. Thus since $|\mathcal{G}_P| > 1$, $\mathcal{G}_P \leq \mathcal{G}_{L_S} = \mathcal{N}_{\mathcal{G}}(S)$. P is not incident to $a(D)$, and $P \neq C(D)$. Thus Lemma 3.11 yields that $K(D) \cap \mathcal{G}_P = \langle 1 \rangle$, and $\mathcal{G}_P \leq S$.

□

The same may not be true for the lines through P_S. Since there is no totally irregular restriction on the lines of Π, there may possibly be an orbit of lines with trivial stabilizer through P_S. In fact this is exactly where the duals of later proofs break down. The existence of a s-center, but no s-axis, will not give an upper bound to the number of lines through a point.

Lemma 4.10 *There are q d-axes through P_S, and possibly q more lines fixed by $K(D)$ for $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$. If a line through P_S has a non-trivial stabilizer, then its stabilizer is a subgroup of S.*

Proof: The proof is similar to the previous one after adding the restriction of a non-trivial stabilizer to the line.

□

In the next lemma, the possibilities for the intersection of a d-axis $a(D)$ with a s-axis L_S are determined when $K(D) \not\leq \mathcal{N}_{\mathcal{G}}(S)$. This lemma is important in pinning down the structure of points on a d-axis when $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line. If $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line, either $p = 3$, or there are $(q-1)/2$ d-centers on a d-axis.

Lemma 4.11 *Assume the existence of the s-axes. Then there are $(q-1)/2$ d-centers on a d-axis, and L_S intersects $a(D)$ in a d-center when $K(D) \not\leq \mathcal{N}_{\mathcal{G}}(S)$; or $p = 3$, there are 0 or 1 d-centers on $a(D)$, and if $K(D) \not\leq \mathcal{N}_{\mathcal{G}}(S)$, L_S intersects $a(D)$ in a point P where $\mathcal{G}_P \leq S$ and $|\mathcal{G}_P| = 3$. If there are both s-axes and s-centers, then there are $(q-1)/2$ d-centers on a d-axis, and two d-axes intersect in an involutory center, d-center or s-center; two d-centers are joined by an involutory axis, d-axis, or a s-axis.*

Proof: By Lemma 3.14, there are three possibilities for the layout of points on a d-axis. Assume first that there are $(q-1)/2$ d-centers incident to $a(D)$. By Lemma 3.14, $C(D)$ is not one of them. By Lemma 4.7, each d-center has two s-axes incident, so the $(q-1)/2$ d-centers account for $(q-1)$ s-axes. Let L_S be one of them with S its corresponding Sylow p-group, and suppose $C(D_1)$ is the point of intersection.

Lemma 3.9 shows $\mathcal{G}_{C(D_1)} = D_1$. Lemma 4.8 shows that if $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$ then L_S intersects $a(D)$ in a point fixed by $K(D)$. Since $C(D_1) \neq C(D)$, $K(D) \cap D_1 = \langle 1 \rangle$ by Lemma 3.3, and so $K(D) \not\leq \mathcal{N}_{\mathcal{G}}(S)$. So the $(q-1)$ s-axes which intersect $a(D)$ in d-centers correspond to Sylow p-groups which are not normalized by $K(D)$. $K(D)$ normalizes two Sylow-p groups by Result 2.2, and there are $q+1$ Sylow p-groups in total, so the $(q-1)$ s-axes through the d-centers correspond exactly to the Sylow p-groups that are not normalized by $K(D)$. Putting this all together, if $K(D) \not\leq \mathcal{N}_{\mathcal{G}}(S)$, then L_S intersects $a(D)$ in a d-center.

Now assume furthermore the existence of s-centers. There are then two s-centers on a d-axis $a(D)$, each incident to $q-1$ other d-axes (Lemma 4.7). Using Lemmas 3.18 and 3.21, $a(D)$ intersects the other d-axes in involutory centers, d-centers, and s-centers, since there are no singlets on $a(D)$ in this case. Now consider the remaining cases where there is at most one d-center $C(D)$ on $a(D)$. These cases will be shown not to exist except when $p = 3$. Suppose $K(D) \not\leq \mathcal{N}_{\mathcal{G}}(S)$ and let $a(D)$ intersect L_S in point P. If $\mathcal{G}_P$ does not fix L_S there are at least two s-axes incident to P. By Lemma 4.7, s-axes intersect in d-centers, so this forces P to be the d-center $C(D)$, and the two s-axes correspond to Sylow p-groups that are normalized by $K(D)$. But the assumption was that $K(D)$ did not normalize S. So $\mathcal{G}_P$ fixes L_S and by Lemma 4.6, $\mathcal{G}_P \leq \mathcal{N}_{\mathcal{G}}(S)$. $\mathcal{G}_P$ moves $a(D)$ in an orbit of at least two d-axes through P, since $\mathcal{G}_P \cap \mathcal{G}_{a(D)} = \langle 1 \rangle$.

P is not an involutory center, d-center nor a singlet since $2 \not| \; |\mathcal{G}_P|$. Thus Lemma 3.19 applies to show that since $\mathcal{G}_P \cap D$ is a trivial group, $p = 3$ and $|\mathcal{G}_P| = 3$. Moreover, the point orbit of P on $a(D)$ accounts for the intersection of $2(q-1)$ d-axes with $a(D)$. Then, by Lemmas 3.18, 3.14 and 3.17, the d-axes intersect $a(D)$ in involutory centers, singlets, and the point orbit of P on $a(D)$.

Now assume also the existence of the s-centers. Let S_1 be a Sylow p-group such that $K(D) \leq S_1$, with corresponding s-center P_{S_1}. By Lemma 4.7, P_{S_1} is incident to $a(D)$, and to $q-1$ other d-axes. However, P_{S_1} is not in the orbit of P since $\mathcal{G}_{P_{S_1}} = \mathcal{N}_{\mathcal{G}}(S_1)$. P_{S_1} is not an involutory center nor a singlet by Lemma 4.6. So P_{S_1} cannot be the intersection of two d-axes. This contradiction shows that if there are both s-centers and s-axes, then there are $\frac{q-1}{2}$ d-centers on a d-axis.

□

In the case where $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point and no line, there is a third possibility: there is a regular orbit of lines. This then does not allow as much of a restriction on the geometry of Π, and so this assumption does not produce, at least in the same way, the conclusions gained with the assumption of a line fixed by $\mathcal{N}_{\mathcal{G}}(S)$.

The existence of both P_S and L_S forces $(q-1)/2$ d-centers on a d-axis. Now it can further be shown that when P_S is incident to L_S, a Desarguesian subplane of order q made up of the involutory centers and axes, d-centers and axes, and the s-centers and axes is formed.

Theorem 4.2 *If $\mathcal{N}_{\mathcal{G}}(S)$ fixes an incident point-line flag (P_S, L_S) then there is a Desarguesian subplane of order q left invariant by $PSL(2,q)$ in Π.*

Proof: The proposed Desarguesian subplane is formed from the involutory centers and axes, the d-centers and d-axes, and the s-centers and s-axes. There are $q(q-1)/2$ involutory centers (axes), $q(q+1)/2$ d-centers (axes), and $q+1$ s-centers (axes), giving a total of q^2+q+1 points and q^2+q+1 lines. It will be shown that this set of points and lines is a closed configuration, forming a subplane of order q. Since the involutory centers, d-centers, and s-centers each form an orbit under $PSL(2,q)$, as do the involutory axes, d-axes, and s-axes, the subplane formed will be invariant under $PSL(2,q)$.

By Lemma 4.11, the existence of a s-center and a s-axis implies there $(q-1)/2$ d-centers on a d-axis, and, by Lemma 3.14, there are also $(q-1)/2$ d-axes through a d-center. By Lemma 3.13, two involutory axes intersect in an involutory center or d-center, and two involutory centers are joined by an involutory axis or d-axis. By Lemma 4.7, two s-axes intersect in a d-centers; two s-centers are joined by a d-axis. Lemma 4.7 also shows that an involutory axis intersects a s-axis in a d-center, and that an involutory center is joined to a s-center by a d-axis. In the proof of Lemma 3.14, it was shown that an involutory axis intersects a d-axis in an involutory center or d-center, and that an involutory center is joined to a d-center by an involutory axis or a d-axis. Lemma 4.11 shows that two d-axes intersect in an involutory center, d-center, or a s-center, and that two d-centers are joined by an involutory axis, d-axis, or a s-axis.

It remains to be seen how d-axes intersect s-axes, and dually, how d-centers are joined to s-centers. Let $a(D)$ be a d-axis. Each of the $(q-1)/2$ d-centers on $a(D)$ has two s-axes incident (Lemma 4.7), so the $(q-1)/2$ d-centers on $a(D)$ account for the intersection of $q-1$ s-axes with $a(D)$. Also by Lemma 4.7 there are two s-centers on $a(D)$. Since P_S I L_S, these two s-centers account for the remaining two s-axes to intersect $a(D)$. So a d-axis intersects a s-axis in either a d-center or a s-center. Similarly, a d-center is joined to a s-center by either a d-axis or a s-axis.

Thus the centers and axes form a closed configuration, and since there exist four points, no three collinear, in this set, this configuration is a subplane of order q. Luneberg has shown that a plane of order q with the group $PSL(2,q)$ acting on it is Desarguesian (Theorem 2.2).

□

4.4 The p-element as Generalized Perspectivity

Assume an element of order p is a generalized perspectivity. Then either all elements of order p are generalized homologies, or all elements of order p are generalized elations. In both cases there is a point or a line fixed by $\mathcal{N}_{\mathcal{G}}(S)$, so the results from the previous section apply. It is shown that no element of order p is a generalized homology. It is possibile for an element of order p to be a generalized elation; either Π is the Desarguesian plane of order q, $n = q^2$ and there is a Desarguesian subplane of order q left invariant by $PSL(2, q)$; or $p = 3$ and $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point but not a line.

Theorem 4.3 *No element of order p is a generalized homology.*

Proof: Since the case where an element of order p is triangular has been ruled out (Theorem 4.1), assume that the element of order p is a non-triangular generalized homology. The existence of one element of order p which is a generalized homology forces all elements of order p to be generalized homologies by Lemma 4.1. Let S be a Sylow p-group. First, it will be shown that the non-trivial elements of S have a common center, which will be denoted $C(S)$, and a common axis, $a(S)$.

If S is of prime order then the group shares the same generalized center and axis. If S is not of prime order, then S is of order p^r where $r > 1$. For $x \in S$, elements of the group $\langle x \rangle$ share the same generalized center and axis. Let x and y be elements in S where $y \notin \langle x \rangle$. Suppose $C(x) \neq C(y)$. Then distinct subgroups of order p in $\langle x, y \rangle$ also have distinct centers, and by Lemma 2.1, have distinct axes. Since $p \geq 3$, and there are $(p^2 - 1)/(p - 1) = p + 1$ distinct subgroups of order p, there are at least 4 distinct subgroups of order p. Let x, y, z, and w be non-trivial representatives from these distinct subgroups. Since they all commute, $C(x)$ is incident to $a(y)$, $a(z)$ and $a(w)$. $C(y)$ is also incident to $a(z)$ and $a(w)$, which forces $C(x) = C(y)$. This contradiction forces $C(x) = C(y)$ and also $a(x) = a(y)$. So all elements in S have the same center $C(S)$ and axis $a(S)$.

Since $C(S)$ and $a(S)$ are unique in $\mathcal{F}(S)$, and $\mathcal{N}_{\mathcal{G}}(S)$ permutes the fixed points and lines of S, $\mathcal{N}_{\mathcal{G}}(S)$ will fix $C(S)$ and $a(S)$. Then the results from the previous section will apply with $C(S) = P_S$ and $a(S) = L_S$. In particular by Lemma 4.11, there will be $(q-1)/2$ d-centers on a d-axis.

Now geometry is used to find number restrictions on n. These together with the upper bound found in Theorem 3.2 will yield a contradiction.

Look at the line $a(S)$. $\mathcal{N}_{\mathcal{G}}(S)$ fixes $a(S)$, and hence permutes the points of $a(S)$. By Lemma 4.7, if $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$, then $C(D)$ is incident to $a(S)$. $C(D)$ is not incident to $a(D)$ by Lemma 3.14, and $C(S)$ is not incident to $a(S)$, so Lemma 4.8 shows that $a(D)$ intersects $a(S)$ in a d-singpoint P_D. Fixed points of a subgroup of $K(D)$ are $C(D)$ or are incident to $a(D)$ by Lemma 3.11, so $K(D)$ fixes exactly two points on $a(S)$, $C(D)$ and P_D, and permutes the rest in semi-regular orbits. This means that $n+1 \equiv 2 \pmod{\frac{q-1}{2}}$; so

$$n \equiv 1 \pmod{\frac{q-1}{2}}. \tag{4.1}$$

There are q $K(D)$s that normalize S, so there are q d-centers and q d-singpoints on $a(S)$; by Lemma 4.9, any remaining points on $a(S)$ have as stabilizers subgroups of S. Since an element x of order p in S is a generalized homology with axis $a(S)$, the fixed points of x are $C(S)$ or incident to $a(S)$. Thus the number of p-points fixed by subgroups of S is $1 + (n+1-2q)$; the total number of p-points, N_P, equals $(q+1)(1+(n+1-2q))$.

The number of d-singpoints on each d-axis, n_d, is now counted. By Lemma 4.11, a d-axis $a(D)$ intersects other d-axes in involutory centers, d-centers, and s-centers.

There are no singlets on $a(D)$, and by Lemma 3.21, the two s-centers are the only p-points on $a(D)$. Thus the remaining points are d-singpoints. There are $\frac{q-1}{2}$ involutory centers on $a(D)$, so that means

$$n_d = (n+1) - \frac{q-1}{2} - \frac{q-1}{2} - 2 = n - q.$$

Putting N_P and n_d into the equation in conclusion (1) of Theorem 3.1 yields

$$n^2+n+1 = q^2 + q\cdot\frac{q-1}{2}(n-q) + \frac{1}{2}q(q+1)(n-q) + (q+1)(n+2-2q). \quad (4.2)$$

Looking at this equation mod $\frac{q+1}{2}$ gives

$$n^2+n+1 \equiv (-1)(-1) + (-1)(-1)(n-q) + 0 + 0 \pmod{\frac{q+1}{2}}$$

and so

$$n^2 \equiv 1 \pmod{\frac{q+1}{2}}. \quad (4.3)$$

Assume first that $n+1 = 2q$. This fits the congruence in 4.1; when put into the congruence of 4.3, the result is

$$(2q-1)^2 \equiv 1 \pmod{\frac{q+1}{2}} \text{ or } (2(-1)-1)^2 \equiv 1 \pmod{\frac{q+1}{2}}.$$

This then means $\frac{q+1}{2}$ divides 8, which implies $q = 3, 7$ or 15. So $q = 7$ and $n = 13$. Putting these values of q and n into equation 4.2 gives

$$(13)^2 + 13 + 1 = 49 + 21(14-8) + 28(14-8) + 8(1)$$

which is a contradiction.

So $n+1 > 2q$. Let P be a point on $a(S)$ which is not a d-center nor incident to a d-axis. Then by Lemma 4.9, $\mathcal{G}_P \leq S$, and the orbit of P on $a(S)$ under S is of length $\frac{|S|}{|\mathcal{G}_P|} = p^l$ where $l \geq 0$. The line $\overline{C(S)P}$ is fixed by $\mathcal{G}_P$ since $\mathcal{G}_P$ fixes P and $C(S)$. The order of $\mathcal{G}_P$ is p^u for $u > 0$, so the points on $\overline{C(S)P}$ which are not $C(S)$ or P

are permuted in orbits of length p^u. Thus $n+1 \equiv 2 \pmod{p}$, so $n \equiv 1 \pmod{p}$. Now look at the orbits of points on $a(S)$ under S: there are two orbits of length q, and the remaining points are in orbits of length p^{l_i} where $l_i \geq 0$. If $l_i > 0$ for all i, then $n+1 \equiv 0 \pmod{p}$, so there must be some point orbit of length 1. This then forces the stabilizer of that point, call it Q, to be S. Now the line $\overline{C(S)Q}$ is fixed by S, and S fixes exactly two points on that line, so $n \equiv 1 \pmod{q}$.

Putting this equivalence together with 4.1 gives $n \equiv 1 \pmod{\frac{q(q-1)}{2}}$. Since there is an involutory homology, $n \equiv 1 \pmod{2}$, and since $q \equiv 3 \pmod{4}$, $\frac{q(q-1)}{2}$ is odd, so putting this all together gives $n \equiv 1 \pmod{q(q-1)}$. So $n = 1 + tq(q-1)$ for $t > 0$. By Theorem 3.2, there is an upper bound to n of $\frac{3}{2}q^2$. Since $1+tq(q-1) \geq \frac{3}{2}q^2$ for $t \geq 2$ and $q \geq 7$, $t = 1$ and $n = q^2 - q + 1$. Now put this together with $n^2 \equiv 1 \pmod{\frac{q+1}{2}}$ from congruence 4.3. Since $n \equiv q^2 - q + 1 \equiv 1-(-1)+1 \equiv 3 \pmod{\frac{q+1}{2}}$, $n^2 \equiv 9 \pmod{\frac{q+1}{2}}$. So $1 \equiv 9 \pmod{\frac{q+1}{2}}$, and $\frac{q+1}{2}$ divides 8, which implies $q = 7$ and $n = 43$.

Putting these data into equation 4.2 gives

$$43^2 + 43 + 1 = 49 + 21(44-8) + 28(44-8) + 8(45-14)$$

which is false. This final contradiction proves no element of order p is a generalized homology.

□

Theorem 4.4 *Assume an element of order p is a generalized elation. If $p \neq 3$, then either $n = q$ or $n = q^2$. The case where $n = q$ occurs if and only if Π is a Desarguesian plane of order q. If $n = q^2$, then $PSL(2,q)$ leaves invariant a Desarguesian subplane of order q, and Π is not Desarguesian. If $p = 3$, then each Sylow p-group has a common axis, or each Sylow-p group has a common center; if there is a common* *axis, then* $n = q^2$ *and there is a Desarguesian subplane of order* q *left invariant by* $PSL(2, q)$.

Proof: The case where $p = 3$ is special and calls for much more work than the other values of p to get even partial results. This proof will be done in steps, which are first outlined here. The assumption that the element of order p is a generalized elation forces there to be either a common center or a common axis for the non-trivial elements of a Sylow p-group S. If $p \neq 3$, then there is both a common center and a common axis for the elements of S. If $p = 3$, and there is a common axis for S, then there is a point P_S on $a(S)$ fixed by $\mathcal{N}_{\mathcal{G}}(S)$. This is the complicated step. This point may or may not be a common center for the elements of S. In both of these cases, then, there is an incident point/line flag fixed by $\mathcal{N}_{\mathcal{G}}(S)$, and so by Theorem 4.2, a Desarguesian subplane of order q left invariant by $PSL(2, q)$ in Π. This Desarguesian subplane is either Π, or $n = q^2$ and Π is not Desarguesian.

Step 1 *There is either a common center or a common axis for the non-trivial elements of* S.

Let x and y be non-trivial elements of S. Suppose that $C(x) \neq C(y)$. Let $1 \neq z \in S$. Then since x fixes $a(z)$, either $a(z)$ is incident with $C(x)$, or else $a(z) = a(x)$, and hence is still incident to $C(x)$. The same is true for y: $a(z)$ is incident to $C(y)$. So for all non-trivial elements z in S, $a(z) = \overline{C(x)C(y)}$, and hence there is a common axis for the non-trivial elements of S. Similarly, the assumption of distinct axes proves the existence of a common center.

Step 2 *If* $p \neq 3$ *there is a common center and a common axis for the non-trivial elements of* S.

Assume $p \neq 3$. By step 1, S has either a common axis or a common center. Suppose it has a common axis. Then all of the centers for elements of S lie on that axis. Let $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$. By Lemma 4.1, $K(D)$ has a cyclic subgroup M of order $(p-1)/2$ which normalizes $\langle s \rangle$ for $s \in S$. Since $p \neq 3$, $M \neq \langle 1 \rangle$, so M fixes each $C(s)$, forcing them all to lie on $a(D)$. Since they also lie on $a(S)$, and $a(S) \neq a(D)$, there must be a common center for the elements of S.

Similarly, the assumption of a common center forces a common axis.

Step 3 *If* $p = 3$ *and there is a common axis* $a(S)$ *for* S *then there is a point* P_S *fixed by* $\mathcal{N}_{\mathcal{G}}(S)$ *and incident to* $a(S)$.

Assume $p = 3$ and there is a common axis for the elements of S. By way of contradiction assume that $\mathcal{N}_{\mathcal{G}}(S)$ does not fix a point.

Since there is no P_S, there can be no common center $C(S)$ for the elements of S. As will be seen, this forces distinct subgroups S of order p to have distinct centers. Suppose not, so that $C(x) = C(y)$ where $\langle x \rangle \neq \langle y \rangle$ for $1 \neq x, y \in S$. Let H be the subgroup of S of elements which have $C(x)$ as a center. By Lemma 4.4, $\mathcal{N}_{\mathcal{G}}(H) = S \cdot T$, where T is a cyclic group of order $(|H|-1)/2$ and $T \leq K(D)$ for some $K(D)$ in $\mathcal{N}_{\mathcal{G}}(S)$. $|H| \geq 9$ as $\langle x, y \rangle \leq H$, so T is not the identity subgroup. $\mathcal{N}_{\mathcal{G}}(H)$ fixes $C(x)$, and in particular $C(x)$ is a fixed point of T. $C(x) \neq C(D)$ so $C(x)$ is incident to $a(D)$ by Lemma 3.11. $K(D)$ moves $C(x)$ in an orbit of length $|K(D)|/|T| = (q-1)/(|H|-1)$ along $a(D)$, and since $H < S$, there will be at least two points in the orbit. This orbit of points consists of centers of elements of S since $K(D)$ permutes the elements of S. The non-trivial elements of S are generalized elations, and their centers lie on the common axis $a(S)$. But $a(S) \neq a(D)$, so this contradiction forces distinct subgroups of order p of S to have distinct centers. It's interesting to note that T is then the identity group, and so the centers are not forced onto $a(D)$.

The distinctness of centers forces $n \equiv 0 \pmod{\frac{q}{3}}$. This is done by counting the lines through a center $C(x)$ where $x \in S$ for S a Sylow p-group. The lines fixed by x

are all incident to $C(x)$ since x is a generalized elation. If $1 \neq y \in S - \langle x \rangle$, then the lines fixed by y are incident to $C(y)$ with $a(S)$ the only line that both x and y fix. S fixes $C(x)$ and so permutes the lines through $C(x)$. S also permutes the lines fixed by x, so S acts separately on two sets of lines through $C(x)$: the ones fixed by x, and the others. For the lines fixed by x, S fixes one line and permutes the rest in cycles of length $q/3$. The rest of the lines through $C(x)$ are permuted by S in cycles of length q. Putting this together gives that $n + 1 \equiv 1 \pmod{\frac{q}{3}}$ and so $n \equiv 0 \pmod{\frac{q}{3}}$.

Now it will be shown that there are $\frac{q-1}{2}$ d-centers on a d-axis; this structure will force a point fixed by $\mathcal{N}_{\mathcal{G}}(S)$ incident to L_S.

So first assume there is at most one d-center on a d-axis, so that either case (1) or case (2) of Lemma 3.14 apply. The equation in conclusion (1) of Theorem 3.1 will be put together with the congruence $n \equiv 0 \pmod{\frac{q}{3}}$ to come up with contradictions.

A count must be done on the number of d-singpoints, n_d, on a d-axis. By Lemma 4.11, the $q - 1$ s-axes, corresponding to Sylow p-groups not normalized by $K(D)$, intersect $a(D)$ in points whose stabilizer is of order 3. Since these points are not d-centers, and s-axes intersect in d-centers, there must be $q-1$ of these points on $a(D)$. By Lemma 3.21, these points are the only p-points on $a(D)$ since they form one point orbit under D. The remaining points on $a(D)$ are involutory centers, singlets, d-centers, and d-singpoints by Lemma 3.20. For case (1) of Lemma 3.14, each d-axis is incident with exactly one d-center, $\frac{1}{2}(q-1)$ involutory centers, $\frac{1}{4}(q-1)(q-3)$ singlets, and $q-1$ p-points, leaving $n_d = n - \frac{1}{4}(q+3)(q-1)$ d-singpoints on $a(D)$. For case (2) of Lemma 3.21, $n_d = n + 1 - \frac{1}{4}(q+5)(q-1)$.

Now to determine N_p for the two cases. The p-points lie on the s-axes since the element of order p in S is a generalized elation with common axis $a(S)$. By Lemma 4.9, there are q d-centers $C(D)$ on $a(S)$, one for each $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$. In case (1) of Lemma 3.14, $C(D)$ is incident to $a(D)$, and so, by Lemma 4.9, the remaining points on $a(S)$ have stabilizers which are non-trivial subgroups of S. Thus there are $n+1-q$ p-points on $a(S)$, and $N_p = (q+1)(n+1-q)$. For case (2) of Lemma 3.14, $C(D)$ is not incident to $a(D)$. By Lemma 4.8, if $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$, then $a(D)$ intersects $a(S)$ in a point whose stabilizer is $K(D)$, i.e. a d-singpoint. Thus there are q d-singpoints on $a(S)$, and q d-centers, leaving $n+1-2q$ p-points on $a(S)$. For case (2) of Lemma 3.14, $N_p = (q+1)(n+1-2q)$.

Now put these numbers into the equation in conclusion (1) of Theorem 3.1. First look at case (1) of Lemma 3.14.

Case 1 $C(D)\ I\ a(D)$

$$n^2+n+1 = q^2+\frac{1}{2}q(q-1)(n-q)+\frac{1}{2}q(q+1)(n-\frac{1}{4}(q+3)(q-1))+(q+1)((n+1)-q)$$

Simplifying this gives

$$n^2-(q^2+q)n \quad = \quad -q\cdot\frac{q-1}{2}(\frac{1}{4}(q^2+8q+3)) \tag{4.4}$$

For $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$, $K(D)$ permutes the points of $a(S)$ fixing only $C(D)$. Thus $n \equiv 0 \pmod{\frac{q-1}{2}}$. Putting this together with $n \equiv 0 \pmod{\frac{q}{3}}$ gives that $n \equiv 0 \pmod{(\frac{q}{3})(\frac{q-1}{2})}$.

Let $n = t\cdot(\frac{q}{3})(\frac{q-1}{2})$. Putting this into equation 4.4 gives

$$(\frac{q-1}{2})^2(\frac{q}{3})^2t^2-(q(q+1)\frac{q}{3}\cdot\frac{q-1}{2}t) = -q\cdot\frac{q-1}{2}(\frac{1}{4}(q^2+8q+3)).$$

By Lemma 3.1, q is an odd power of 3, so $q \geq 27$. Cancelling $q\cdot\frac{q-1}{2}$ from both sides gives

$$\frac{q-1}{2}\cdot\frac{q}{9}\cdot t^2-(q+1)\frac{q}{3}t = -\frac{1}{4}(q^2+8q+3).$$

Let $q > 27$. Since $\frac{q}{9}$ is divisible by 9 and 4 is a unit mod 9, the equation is $0 \equiv -\frac{1}{4}(0+0+3) \pmod 9$ which is a contradiction.

If $q = 27$, the equation is

$$13 \cdot 3t^2 - 28 \cdot 9t = -\frac{1}{4}(27^2 + 8 \cdot 27 + 3).$$

This has no integer solution, so this final contradiction shows that $C(D)$ is not incident to $a(D)$.

Case 2 *$C(D)$ is not incident to $a(D)$*

Now the equation from conclusion (1) of Theorem 3.1 yields

$$n^2+n+1 = q^2+\frac{1}{2}q(q-1)(n-q)+\frac{1}{2}q(q+1)(n+1)-\frac{1}{4}(q+5)(q-1)+(q+1)((n+1)-2q)$$

This then gives that

$$n^2 - (q^2 + q)n \;=\; -q^2\frac{q-1}{2} + q\frac{q+1}{2} - q\left(\frac{q+1}{2}\right)\left(\frac{q+5}{2}\right)\left(\frac{q-1}{2}\right) - q(q-1).$$

Since $n \equiv 0 \pmod{\frac{q}{3}}$, $n = \frac{q}{3}t$ for some t. Putting this into the equation gives:

$$\frac{q^2}{9}t^2 - (q^2 + q)\frac{q}{3}t = -q^2 \cdot \frac{q-1}{2} + q \cdot \frac{q+1}{2} - q \cdot \frac{q+1}{2} \cdot \frac{q+5}{2} \cdot \frac{q-1}{2} - q(q-1).$$

Cancelling out a q on both sides gives:

$$\frac{q}{9}t^2 - (q+1)\frac{q}{3}t = -q \cdot \frac{q-1}{2} + \frac{q+1}{2} - (\frac{q+1}{2})(\frac{q+5}{2})(\frac{q-1}{2}) - (q-1).$$

Looking at this equation mod 3, and using $q \equiv 0 \pmod 3$ and $q \geq 27$ gives

$$0 \;\equiv\; 0 + \frac{1}{2} - (\frac{1}{2})(\frac{5}{2})(\frac{-1}{2}) - (-1) \equiv 1 \pmod 3.$$

This contradiction shows that case (2) of Lemma 3.14 also does not hold, so that leaves case (3) where there are $\frac{q-1}{2}$ d-centers on a d-axis. It will now be shown that there is a point P_S incident to $a(S)$.

Let $a(D)$ be a d-axis. By Lemma 4.7, each of the d-centers $C(D_i)$ on $a(D)$ have two s-axes corresponding to Sylow p-groups normalized by $K(D_i)$. None of these s-axes correspond to Sylow p-groups that $K(D)$ normalizes, because by Lemma 4.8, if $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$, then $a(S)$ intersects $a(D)$ in a point fixed by $K(D)$. The only d-center fixed by $K(D)$ is $C(D)$, which is not incident to $a(D)$. So the d-centers on $a(D)$ have $q-1$ s-axes incident, and there are two more s-axes which correspond to Sylow p-groups normalized by $K(D)$, and which intersect $a(D)$ in points fixed by $K(D)$.

By Lemma 3.21, there are p-points on $a(D)$. Let P be one of these points and suppose x is an element of order p that fixes P. Then P is incident to $a(S)$ where $x \in S$. Since P is not a d-center, then the proof of Lemma 4.11 shows that $a(S)$ must be one of the s-axes such that $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$; thus P is fixed by $K(D)$ by Lemma 4.8. So $\langle K(D), x\rangle \leq \mathcal{G}_P$. By Lemma 3.15, G_P is a semi-direct product of an elementary abelian p-group of order p^m with a cyclic group of order t where t is a divisor of $\frac{(p^{(m,r)}-1)}{2}$ and $\mathcal{G}_P \leq \mathcal{N}_{\mathcal{G}}(S)$. Since t in this case equals $(p^r-1)/2$, $m = r$ and $\mathcal{G}_P = \mathcal{N}_{\mathcal{G}}(S)$. Thus there is a point fixed by $\mathcal{N}_{\mathcal{G}}(S)$ which lies on $a(S)$.

Step 4 *If $p \neq 3$, or if $p = 3$ and there is a common axis, then there is a Desarguesian subplane of order q left invariant by $PSL(2,q)$ and n is q or q^2. The case where $n = q$ occurs if and only if Π is a Desarguesian plane of order q. In the case where $n = q^2$, Π is not Desarguesian*

If $p \neq 3$, then by step 2 there is a common center and axis, $C(S)$ and $a(S)$ respectively, for a Sylow p-group S. If $p = 3$ and there is a common axis $a(S)$ for a Sylow p-group S, then by step 3 there is a point P_S fixed by $\mathcal{N}_{\mathcal{G}}(S)$. So in both of these two cases, $\mathcal{N}_{\mathcal{G}}(S)$ fixes an incident point/line flag and so Theorem 4.2 applies to show that there is a Desarguesian subplane Π_o of order q in Π left invariant by $PSL(2,q)$. If $n = q$, then Theorem 2.2 applies to show that Π is a Desarguesian subplane of order q, and it is known that the Desarguesian plane of order q does admit $PSL(2,q)$ acting totally irregularly with an involutory homology.

Now assume that $n > q$. By Lemmas 3.7 and 3.11, the fixed points of elements of order relatively prime to p lie on either d-axes or involutory axes. The fixed points of elements of S lie on $a(S)$. Therefore all of the points of the plane Π lie on one of the lines of Π_o. Thus Π_o is a Baer subplane, and so $n = q^2$ by Theorem 2.3.

Suppose that Π is Desarguesian. Then $PSL(2,q)$ is a subgroup of the collineation group of Π; in fact $PSL(2,q) \leq PSL(3,q^2)$. Then the elements of $PSL(2,q)$ can be represented by matrices in their action on Π, and thus an element of $PSL(2,q)$ is triangular, or is a flag or anti-flag collineation, or a perspectivity. Since an element $1 \neq x \in S < PSL(2,q)$ is a generalized elation with generalized center $C(S)$ incident to $a(S)$, and by Lemma 3.22, x is not an elation, x must be a flag collineation. This means every element of order p in S is a flag collineation, fixing only $C(S)$ and $a(S)$. By Lemma 4.9, there are at least $q^2 - 2q$ points on $a(S)$ which are fixed by non-trivial subgroups of S. This means some element of order p in S must fix some point on $a(S)$ other than $C(S)$; but then x is not a flag collineation. This contradiction proves that Π is not Desarguesian.

□

Does a plane Π of order q^2 exist which satisfies the conditions: $PSL(2,q)$ is a totally irregular collineation group with an involutory homology, and $PSL(2,q)$ leaves invariant a Desarguesian subplane of order q. Searching through the literature gleaned no results. $PSL(2,q)$ fixes no point or line, and the only known planes where the collineation group does not fix a point and line are the Desarguesian, Hughes, and Figueroa planes.

4.5 A Planar p-element

Now assume the element of order p is planar. This case is complicated; there can be different subplanes associated with different size subgroups of S. In Lemma 4.2 it was proved that at most one point and line are fixed by $\mathcal{N}_{\mathcal{G}}(S)$. In this section, it is shown that if $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line, no element of order p is planar. The line is forced to be in every subplane and can be used to establish an upper bound for n. A lower bound is found, and a contradiction ensues. Unfortunately the same idea does not work when the assumption is that $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point P_S. The assumption of a totally irregular action means each point has a non-trivial stabilizer but not necessarily that every line does. There may be an orbit of lines through P_S with trivial stabilizers, so an upper bound for n cannot be achieved in the same way as when $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line.

There are limited results for the remaining cases: $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point, and $\mathcal{N}_{\mathcal{G}}(S)$ fixes no point and no line. If q is prime and $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point, then no element of order p is planar. If q is prime then there is a bound on the order of the subplane. More results were found which are not included in this work but do provide a path for future research.

Theorem 4.5 *Suppose that $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line. Then no element of order p is planar.*

Proof: Suppose that an element of order p is planar. Then all elements of order p are planar by Lemma 4.1. Let the subplane fixed by the element x be denoted Π_x and m_x be the order of Π_x. Since $\mathcal{N}_{\mathcal{G}}(S)$ fixes the line L_S, then in particular $x \in S$ fixes L_S, and L_S is a line of the subplane Π_x. Thus there are $m_x + 1$ points of Π_x on L_S.

Now look at the number of points on L_S. There are q d-centers on L_S and possibly q more points where $a(D)$ intersects L_S for $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$. By Lemma 4.9, the remaining points are fixed by non-trivial elements of S. It is possible that the subplanes Π_x and Π_y have points in common for $x, y \in S$ where $\langle x\rangle \neq \langle y\rangle$. In fact there could be subplanes corresponding to different size subgroups in S. However, if $x \in H$, then $\mathcal{F}(H) \subseteq \mathcal{F}(x)$. Therefore, to get an upper bound on the number of points on L_S fixed by non-trivial elements of S, it is sufficient to count the number of points of Π_x on L_S for each distinct group $\langle x\rangle$ of order p in S.

Each subgroup $\langle x\rangle$ contributes $m_x + 1$ points to L_S; there are $(q-1)/(p-1)$ distinct subgroups of order p in S so

$$n+1 \leq 2q + \frac{q-1}{p-1}(m_x+1). \tag{4.5}$$

By Theorem 3.2, $n < p/(p-1)q^2 \leq \frac{3}{2}q^2$, and no element of order p is Baer by Lemma 3.22, so $m_x < \sqrt{\frac{3}{2}}\,q$ by Lemma 2.3. Since $\sqrt{\frac{3}{2}} < 1.23$, and for $q \geq 7$, $1 < .25q$, it follows that $m_x + 1 < 1.23q + 1 < 1.23q + .25q < 1.5q$.

Now putting this together with equation 4.5 and using that $p \geq 3$,

$$\begin{aligned} n+1 &\leq 2q + (\frac{q-1}{p-1})\frac{3}{2}q \leq 2q + (\frac{q-1}{2})\frac{3}{2}q \\ &= \frac{3}{4}q^2 + \frac{5}{4}q < q^2 \text{ for } q \geq 7. \end{aligned}$$

So $n < q^2$, $m_x < q$ and $m_x + 1 \leq q$.

Putting this new bound on $m_x + 1$ into equation 4.5 gives that

$$n+1 \leq 2q + (\frac{q-1}{p-1})q. \tag{4.6}$$

Assume first that $p \neq 3$. Then equation 4.6 gives that

$$n+1 \leq 2q + (\frac{q-1}{6})q \tag{4.7}$$

since $p \geq 7$. Now we will combine this with a lower bound on $n+1$ to get a contradiction.

The lower bound is found by looking at the number of lines through a particular point P. By Lemma 3.13 involutory centers are joined by either involutory axes or d-axes, so if P is not incident to any d-axis nor to any involutory axis, then it will be joined to the involutory centers of Π by distinct lines. This would mean that there are at least $q(q-1)/2$ lines through P. So the next step is to determine if such a point exists.

The candidate is a point on L_S fixed by $1 \neq x \in S$. There are $m_x + 1 \geq 3$ fixed points of x on L_S. The intersections of the d-axes and involutory axes with L_S will be looked at to show that there exists a point fixed by x which is not incident to any involutory axis or d-axis.

By Lemma 4.11, and since $p \neq 3$ in the present situation, there are $\frac{q-1}{2}$ d-centers on a d-axis, and thus $\frac{q-1}{2}$ d-axes through a d-center. Therefore the q d-centers on L_S account for the intersection of $q(\frac{q-1}{2})$ d-axes with L_S. These d-centers also account for the intersection of all of the involutory axes with L_S. There are q d-axes $a(D)$ remaining, where $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$. By Lemma 4.8, $a(D)$ intersects L_S in a point whose stabilizer is $K(D)$ or in a point fixed by $\mathcal{N}_{\mathcal{G}}(S)$. There is at most one point P_S fixed by $\mathcal{N}_{\mathcal{G}}(S)$, and if it exists it would be incident to all q of the remaining d-axes.

So the points of intersection of involutory axes and d-axes with L_S have been determined: they are either d-centers, points with stabilizers isomorphic to $K(D)$, or P_S. Since the only one of these that x fixes is P_S, and x fixes at least 3 points on L_S, there exists a point P fixed by x on L_S which is not incident to any involutory axis or d-axis. P is joined to the $q(q-1)/2$ involutory centers with distinct lines, and as P is a point in the subplane Π_x, it is incident with $m_x + 1$ lines of Π_x. A line through an involutory center is fixed by the involution, so the line joining P to an involutory center cannot also be a line fixed by an element of order p, and thus cannot be a line of the subplane Π_x. Thus there are at least $(q(q-1)/2) + m_x + 1$ lines through P, so that

$$n+1 \geq q(\frac{q-1}{2}) + m_x + 1 \geq q(\frac{q-1}{2}) + 3. \tag{4.8}$$

Putting this together with equation 4.7 gives:

$$q(\frac{q-1}{2}) + 3 \leq \frac{q^2}{6} + 2q \quad \text{so} \quad 2q^2 - 15q + 18 \leq 0.$$

This is false for $q \geq 7$.

Now let $p = 3$. If there are $(q-1)/2$ d-centers on a d-axis, then equation 4.8 applies and $n+1 \geq q(\frac{q-1}{2}) + 3$. Otherwise, assume there are not $(q-1)/2$ d-centers on a d-axis. In that case, if $K(D) \not\leq \mathcal{N}_{\mathcal{G}}(S)$, L_S intersects $a(D)$ in a point fixed by an element of order 3 (Lemma 4.11), and there are exactly three d-axes through that point. For the purposes of this argument call those points *3d-points*, noting that such a point is characterized by having exactly one s-axis and 3 d-axes incident. The d-axes intersect only in involutory centers, singlets or in 3d-points.

Now look at points of Π_x for some $x \in S$. The idea is to again get a lower bound on the number of lines through a point by choosing a point of Π_x which is not on any involutory axis or d-axis, and hence is joined to the involutory centers by distinct lines. Since Π_x is a non-degenerate subplane there exist points of Π_x not incident to L_S. Let P be one of them. Since $2p \not| \, |\mathcal{G}_P|$ (Lemma 3.15), P is not incident to an involutory axis. So P is not an involutory center, d-center nor a singlet. If P is incident to $L_{\tilde{S}}$ for some Sylow p-group $\tilde{S} \neq S$, then since x does not fix $L_{\tilde{S}}$, x moves it in a line orbit of length three through the point P. However, s-axes intersect in pairs in d-centers, so this cannot be. Since P is not on any s-axis, it is not a 3d-point. So P is not a point where d-axes intersect. If it is incident to just one d-axis $a(D)$, then x fixes $a(D)$, and so by Lemma 3.9, $x \in D$ which is not true. So P is not incident to any d-axis, nor to any involutory axis, and is joined to the involutory centers by $q(q-1)/2$ distinct lines. There are also $m_x + 1$ lines of Π_x

incident to P. These lines are distinct from the ones joining P to involutory centers, so $n+1 \geq q(\frac{q-1}{2}) + m_x + 1 \geq q(\frac{q-1}{2}) + 3$.

Thus when $p = 3$

$$n+1 \geq q\left(\frac{q-1}{2}\right) + 3.$$

Now to pin down the upper bound for the case $p = 3$. Since q is an odd power of p by Lemma 3.1, and $q > 7$, $q \geq 27$. Equation 4.6 gives as an upper bound for the case $p = 3$ and $q \geq 27$

$$\begin{aligned} n+1 &\leq 2q + q(\frac{q-1}{2}) = \frac{q^2}{2} + \frac{3}{2}q \\ &= \frac{q^2}{2} + \frac{1}{18}27q \leq \frac{5}{9}q^2. \end{aligned}$$

So $n < (5/9)q^2$, and $m_x < (\sqrt{5}/3)q < .75q$. This gives that $m_x + 1 < .75q + 1 < .8q$. Now throw this back into equation 4.5 for one more churn through to get

$$n+1 \leq 2q + \frac{q-1}{2}(.8q).$$

Putting this together with the lower bound gives that

$$3 + q\frac{q-1}{2} \leq n+1 < 2q + \frac{q-1}{2}(.8q)$$

which gives $q^2 - 21q + 30 < 0$. Since $q^2 - 21q + 30$ is positive for $q \geq 27$, this gives a contradiction. Thus if a line is fixed by $\mathcal{N}_{\mathcal{G}}(S)$, no element of order p is planar.

□

If $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point the same result does not necessarily follow. A lot of the lemmas that made up the structure for Theorem 4.5 do not necessarily have duals, because the assumption of a totally irregular action on points does not necessarily imply a totally irregular action on lines. However, there are some results when $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point. The next theorem makes an additional restriction that q is prime. In this case the assumption of P_S forces the existence of L_S.

Corollary 4.1 *If $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point, then S is not planar for S a Sylow p-group. In particular if q is prime and $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point, then no element of order p is planar.*

Proof: Assume S is planar, and let m be the order of the plane Π_S. If $K(D) \leq \mathcal{N}_{\mathcal{G}}(S)$, then $K(D)$ is a collineation group of Π_S. By Result 2.4, the point orbits and line orbits of Π_S have the same cyclic structure. If $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point, then $K(D)$ fixes that point, and hence a line of Π_S is fixed by $K(D)$. That line is then fixed by $\mathcal{N}_{\mathcal{G}}(S)$, and Theorem 4.5 now applies to give a contradiction.

□

Lemma 4.12 *Assume $q = p$ and that a Sylow p-group S is planar, with Π_S the subplane $\mathcal{F}(S)$. If m is the order of Π_S, then $m \leq \frac{q-1}{2}$.*

Proof: Let $K(D)$ be a subgroup of $\mathcal{N}_{\mathcal{G}}(S)$. This lemma is proved by looking at the intersections of the lines of Π_S with $a(D)$. It turns out there is a limited number of points where the intersections can occur, which forces the bound on m.

There is at most one point P of Π_S incident to $a(D)$ because, if there were two, $a(D)$ would be fixed by S contradicting Lemma 3.9. Similarly, there can be at most one line of Π_S incident with $C(D)$. If a point P of Π_S is incident to $a(D)$, then $K(D)$ fixes it. Otherwise $K(D)$ would move P in an orbit of length greater than two on $a(D)$, and since $K(D)$ permutes the points of Π_S, this would result in two points of Π_S on $a(D)$. Similarlys, if a line l of Π_S is incident with $C(D)$, then $K(D)$ fixes l.

Next, it will be shown that there is at most one point Q on $a(D)$ such that: Q is not a point of Π_S, $\mathcal{G}_Q \cap \mathrm{K(D)} \neq \langle 1 \rangle$, and Q is incident to a line L of Π_S. Let $H = \mathcal{G}_Q \cap \mathrm{K(D)}$. H permutes the lines of Π_S, and so moves L in a line orbit through Q. If $H \not\leq \mathcal{G}_L$, there are then two lines of Π_S incident to Q, forcing Q to be a point of Π_S. Thus, H fixes L. Then Lemma 4.2 applies to show L is incident to $C(D)$. If there were two such points, Q_1 and Q_2, each incident with necessarily distinct lines of Π_S, then those two lines would intersect in $C(D)$ forcing $C(D)$ to be a point of Π_S. This contradiction proves the assertion.

So the lines of Π_S may be incident to at most two points on $a(D)$ whose stabilizers intersect $K(D)$ non-trivally; one of these may be a point of Π_S. What other points on $a(D)$ can serve as the intersection of lines of Π_S with $a(D)$? By Lemma 3.21, the points on $a(D)$ whose stabilizers intersect $K(D)$ trivially are involutory centers, possibly singlets, and possibly d-centers. No involutory center can have a line l of Π_S incident since otherwise the involution would fix l. At most one line of Π_S can be incident to a singlet or d-center, because otherwise the point would be a point of Π_S with stabilizer divisible by $2p$ contradicting Lemma 3.15. The maximum number of singlets and/or d-centers on $a(D)$ is $(\frac{q-1}{2})^2$ by Lemma 3.14, so these points account for at most $(\frac{q-1}{2})^2$ lines of Π_S.

So now a count is done of the lines of Π_S in terms of possible intersections with $a(D)$. There is at most one point of Π_S incident with $m+1$ lines, at most $(\frac{q-1}{2})^2$ singlets or d-centers incident with at most one line each, and at most one other point with at most one line of Π_S incident. Thus

$$m^2 + m + 1 \leq (\frac{q-1}{2})^2 + (m+1) + 1$$

which gives the result.

□

4.6 Proofs of Theorems 1.1 and 1.2

The geometry due to the involutory homologies has been set up, and it has been established that points fixed by non-trivial elements prime to p are incident to d-axes and involutory axes. The geometries induced by different possibilities for $\mathcal{F}(x)$ where x is an element of order p have been examined. Theorem 1.1 summarizes these results, indicating what possibilities remain, together with restrictions on n and the structure of Π.

Proof of Theorem 1.1

It has been shown that a finite projective plane of order n admits $PSL(2,3)$ as a totally irregular collineation group if and only if the plane is Desarguesian and $n = 2$, 3, or 4 [9]. With the added assumption of an involutory homology this forces $n = 3$.

Now assume $q \geq 7$. By Theorem 3.2, $n < \frac{p}{p-1}(q^2 - 1)$.

Totally irregular action forces every point to be fixed by some prime element of $\mathcal{G}$. By the partition property of $PSL(2,q)$(Result 2.2), the non-trivial elements of order prime to p are in cyclic groups of order $\frac{1}{2}(q \pm 1)$, and by Lemmas 3.7 and 3.11, the fixed points of non-trivial elements in the cyclic groups of order $\frac{1}{2}(q \pm 1)$ lie on involutory axes or d-axes. It then remains to consider the structure of $\mathcal{F}(x)$, where x is an element of order p, since Lemma 4.1 shows that all elements of order p have equivalent fixed point/line substructures.

The element x may be fixed-point-free, triangular, or planar, or it may be a generalized perspectivity. By Theorem 4.1, no element of order p is triangular, and Theorem 4.3 shows that no element of order p is a generalized homology.

If the element of order p is a generalized elation, then by Theorem 4.4 either $n = q$; $n = q^2$, and there is a Desarguesian subplane of order q left invariant by $PSL(2,q)$; or $p = 3$. In the cases where n is q or q^2, Theorem 4.4 gives the conclusions of statement (1). If $p = 3$, then by Theorem 4.4, there is either a common axis or a common center for the elements of a Sylow p-group S. If $p = 3$ and there is a common axis for the elements of a Sylow-p group, then n is either q or q^2 and the conclusions of statement (1) hold. If there is a common center then by the same argument as in step 3 of Theorem 4.4, there must be distinct axes for distinct subgroups of order p in S. This then is statement (2) of Theorem 1.1.

If x is fixed-point-free, then Lemma 3.4 applies to show either $n = (\frac{q^2-1}{2}) \pm \sqrt{\frac{1}{8}(q^4 - 3q^3 + 9q^2 - q - 6)}$ or $n = (\frac{q^2-1}{2}) \pm \sqrt{\frac{1}{8}(q^4 - 5q^3 + 13q^2 + 5q - 6)}$. If it is also assumed that the stabilizer of every point is even, then $q = 7$ and $n = 9$ or 11, and the planes are Desarguesian. This situation does exist: $PSL(2,7)$ contains an involutory homology and acts totally irregularly on the Desarguesian planes of order 9 and 11. This then is statement (3).

The remaining case is where x is planar. Theorem 4.5 rules out this case when $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line for S a Sylow p-group. Corollary 4.1 rules out this case when $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point and S is planar. By Lemma 4.2 at most one point and/or line is fixed by $\mathcal{N}_{\mathcal{G}}(S)$, so the remaining cases are when $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point and S is not planar (statement (4)), or when $\mathcal{N}_{\mathcal{G}}(S)$ fixes no point and no line (statement 5).

□

By Theorem 3.3, $PSL(2,q)$ either leaves invariant a subplane of order q or else acts strongly irreducibly. If the element of order p is a generalized elation, Theorem 4.4 showed that, except possibly when $p = 3$, $PSL(2,q)$ leaves invariant a subplane of order q, and the order of Π is q or q^2. This next theorem goes in the other direction: if a subplane of order q is left invariant by $PSL(2,q)$, then the element of order p is a generalized elation.

Proof of Theorem 1.2

By Theorem 3.3 if $PSL(2,q)$ does not act strongly irreducibly on Π then $PSL(2,q)$ leaves invariant a Desarguesian subplane Π_o of order q. The action of $PSL(2,q)$ on a Desarguesian plane of order q is well known: in particular $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line of the subplane, and hence a line in Π.

Since $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line, Theorems 4.3, 4.4, and 4.5 apply to show the element of order p must be a generalized elation and that $n = q^2$.

□

CHAPTER 5
$PSL(2,7)$

For this chapter it is assumed that $\mathcal{G}$ =PSL(2,7), and that $\mathcal{G}$ is a totally irregular collineation group of a projective plane Π of order n. Instead of assuming that $\mathcal{G}$ has an involutory homology, the existence of a perspectivity is assumed.

Proof of Theorem 1.3

Using Lemma 2.3 the subgroups of $PSL(2,7)$ are isomorphic to D_8, S_3, A_4, S_4, Z_7 and the Frobenius group of order 21 along with the obvious subgroups of the above. From this it is seen that the elements of $\mathcal{G}$ are of order 2, 3, 4 and 7. Lemma 2.2 shows that there are 21 cyclic subgroups of order 4, each with normalizer isomorphic to D_8, 28 cyclic subgroups of order 3, each with normalizer isomorphic to D_6, and 8 cyclic subgroups of order 7, each with normalizer a Frobenius group of order 21. The cyclic groups of order 3, 4, and 7 are a partition of the non-trivial elements of $\mathcal{G}$. For an involution i, $C_{\mathcal{G}}(i) \cong D_8$; all of the involutions are conjugate.

An element of order 2 can be written as the product of two elements of order 3. This can be seen by looking at A_4. Since the elements of order 2 are conjugate, every element of order 2 is the product of two elements of order 3.

An element of order 2 can also be written as the product of two elements of order 7. One way to see this is to use the character table for $PSL(2,7)$ and compute the structural constant. Also, direct computation shows that the elements $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ are of order 7 and multiply to give $\begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}$ which is of order 2. Again, since all of the elements of order 2 are conjugate, this shows all involutions can be written as the product of two elements of order 7.

Let x be a non-trivial perspectivity of $\mathcal{G}$. It is of order 2, 3, 4, or 7. If it's order is 3 or 7, then Result 2.1 applies to show that the involution is a generalized perspectivity. Since an involution is either planar or is a perspectivity, this forces the involution to be a perspectivity. If x is of order 4, then any power of x is also a perspectivity so the involution is a perspectivity. So the existence of a non-trivial perspectivity forces the involution to be a perspectivity.

If the involution is a homology, then results from the previous chapters apply. Assume first that $n \leq 7$. The possibilities for n are 3, 5, and 7, and the planes are Desarguesian. The Desarguesian plane of order 7 does admit $PSL(2,q)$ acting totally irregularly. Using Bloom's theorem, it is seen that $PSL(2,7)$ is not a subgroup of $PSL(3,q)$ for $q = 3,5$ [1], and so $PSL(2,7)$ does not act on the Desarguesian subplanes of order 3 and 5.

Now assume $n > 7$. Theorem 3.4 applies to show that if the element of order 7 fixes no point, then n could be 9, 11, 37, or 39. For the cases where n is 9 or 11, Π is Desarguesian, and the Desarguesian planes of order 9 and 11 do admit $PSL(2,7)$ as a totally irregular collineation group with involutory homology. Theorem 4.1 shows no element of order 7 is triangular. Theorems 4.4 and 4.3 apply to show that if the element of order 7 is a generalized perspectivity, then $n = 49$ and the element of order 7 is a generalized elation. Theorem 4.5 and Corollary 4.1 apply to show that if $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point or a line, no element of order 7 is planar. There remains the case where the element of order 7 is planar and $\mathcal{N}_{\mathcal{G}}(S)$ fixes no point and no line.

So assume the element of order 7 is planar, m is the order of the subplane, and $\mathcal{N}_{\mathcal{G}}(S)$ fixes no point and no line of the plane. By Lemma 4.12, $m \leq 3$, so m is 2 or 3. Since $\mathcal{N}_{\mathcal{G}}(S)$ fixes no point, the number of fixed points of S is divisible by 3 (Lemma 4.2), and so 3 divides $m^2 + m + 1$. This contradiction proves no element of order 7 is planar.

Now assume the involution is an elation. First it will be established that there is no point or line fixed by $PSL(2,7)$.

Step 1 *No point or line is fixed by* $PSL(2,7)$

Assume first that there is a common center C for $\mathcal{G}$. Then all elements of $\mathcal{G}$ are perspectivities. Let l be the axis of an involution. Since the involutions are elations, l is incident to C. Thus $l^{\mathcal{G}} = l$, and so l is a common axis for the involutions of $\mathcal{G}$, which forces l as a common axis for $\mathcal{G}$. Since C is incident to l, the fixed points of the elements of $\mathcal{G}$ are incident to l. Since every point is fixed by a non-trivial element of $\mathcal{G}$, this means that Π consists precisely of the points on l. This cannot be, so there is no common center for $\mathcal{G}$.

Now assume there is a common axis L. For an involution i, $C(i)$ is incident to L, and so $C(i)^{\mathcal{G}} = C(i)$. This means $C(i)$ is a common center for all of the involutions of $\mathcal{G}$, which forces $C(i)$ to be a common center for $\mathcal{G}$. Since this is false, there is no common center, and no common axis for $\mathcal{G}$.

Now assume that P is a point fixed by $\mathcal{G}$. All of the involutory axes are then incident to P. If P were an involutory center, then P would be a common center for all of the involutions of $\mathcal{G}$, and hence would be a common center for $\mathcal{G}$. So P is not an involutory center. For an involution i, the involutory axes of $\mathcal{C}_{\mathcal{G}}(i)$ are incident to $C(i)$. Since they are also incident to P, and P is not an involutory center, $a(i)$ is a common axis for the involutions of $\mathcal{C}_{\mathcal{G}}(i)$. The centralizers of involutions are conjugate, so each $\mathcal{C}_{\mathcal{G}}(k)$ has a common axis $a(k)$ for k an involution. This means that $a(i)$ is a common axis for $\langle\{\mathcal{C}_{\mathcal{G}}(k)\}\rangle$ where k is an involution in $\mathcal{C}_{\mathcal{G}}(i)$. There are 5 involution in $\mathcal{C}_{\mathcal{G}}(i)$, and by looking at the list of subgroups in Result 2.2 it is seen that the subgroup in $\mathcal{G}$ generated by the centralizers of these 5 involution is $PSL(2,q)$ itself. This then forces a common axis for $\mathcal{G}$ which is a contradiction.

A similar argument also shows no line is fixed by $\mathcal{G}$.

Step 2 *Geometric structure due to involutory elations*

Let j, k, l, and m be the non-central involutions in $\mathcal{C}_{\mathcal{G}}(i)$, where $H = \{j, k, i, e\}$ is one K_4 group in $\mathcal{C}_{\mathcal{G}}(i)$, $M = \{l, m, i, e\}$ is the other, and e is the identity element. The involutions in H commute, and hence, since they are elations, have either a common center or a common axis. The same is true of M. Using the same argument as above, a common center for $\mathcal{C}_{\mathcal{G}}(i)$ forces a common center for $\mathcal{G}$, contradicting step 1. Similarily, $\mathcal{C}_{\mathcal{G}}(i)$ does not have a common axis.

Suppose that H has a common center (necessarily $C(i)$) and a common axis (necessarily $a(i)$). M has either a common center or a common axis; assume M has a common axis. Since $i \in M$, the common axis for M is $a(i)$. But then $a(i)$ is a common axis for $\langle M, H \rangle = \mathcal{C}_{\mathcal{G}}(i)$ which is a contradiction. Similarily, there is a contradiction if M has a common center. Thus H can not have both a common center and a common axis; by a similar argument, neither can M. So one group has a common center and distinct axes, and the other has a common axis and distinct centers. Without loss of generality, assume that H has the common center and distinct axes, and that M has the common axis and distinct centers.

Since the three involutions in H have a common center, and all involutions are conjugate, each involution is in a conjugate of H, sharing a common center with two other involutions in that conjugate. So the 21 involutions are grouped in 3's, sharing common centers; thus, there are at most 7 distinct centers. The only alternative is for one common center for the involutions, but that would force a common center for $\mathcal{G}$. Thus there are exactly 7 distinct centers where each center is a common center for exactly 3 involutions. A similar argument using M shows there are 7 distinct axes, where each axis is a common axis for exactly three involutions which are in a conjugate of M.

Looking at $\mathcal{C}_{\mathcal{G}}(i)$ again, there are three distinct axes, $a(i)$, $a(j)$, and $a(k)$ incident with $C(i)$ ($= C(j) = C(k)$), and three centers $C(m)$, $C(l)$, $C(i)$ incident to $a(i)$ ($= a(m) = a(l)$). Thus each axis has three centers incident, and each center has three axes incident. In particular the three axes through $C(i)$ each have two other centers incident, and so $C(i)$ is joined to the 6 other centers by involutory axes. This implies that any two involutory centers are joined by involutory axes, and any two involutory axes intersect in an involutory center, and thus a (Fano) subplane of order 2 is formed by the involutory centers and axes.

Now it remains to consider the contribution of the other elements of $\mathcal{G}$. $\mathcal{C}_{\mathcal{G}}(i)$ is a dihedral group of order 8, and so the product of j and m is an element x of order 4. Since $C(j) \neq C(m)$ and $a(j) \neq a(m)$, Lemma 2.1 applies to show x is a generalized perspectivity which fixes $a(j) \cap a(m) = C(i)$ and where any other fixed points of x are incident to $\overline{C(j)C(m)} = a(i)$. Each cyclic group of order 4 is in precisely one dihedral group of order 8, and all of the dihedral groups of order 8 are conjugate, so each element of order 4 fixes an involutory center, and its remaining fixed points are incident to an involutory axis.

To determine the location of the fixed points of an element of order 3, consider $\mathcal{N}_{\mathcal{G}}(M)$. Since $\mathcal{N}_{\mathcal{G}}(M) \cong S_4$, a group isomorphic to S_4 will fix the common axis of M, $a(i) = a(m) = a(l)$. Since $\mathcal{C}_{\mathcal{G}}(m) < \mathcal{N}_{\mathcal{G}}(M)$, there are two other involutions p and n in $\mathcal{N}_{\mathcal{G}}(M)$, where p and n are distinct from l, i and m, and p and n centralize m. Since m, n and p do not share a common axis, they must have a common center $C(m) = C(p) = C(n)$. The product jp is an element of order 3. $C(p) \neq C(j)$ and $a(p) \neq a(j)$, so the element jp of order 3 is a generalized perspectivity fixing $a(j) \cap a(p)$, and any other fixed points are incident to $\overline{C(j)C(p)} = a(i)$. The subgroups of order 3 are conjugate, so the fixed points of all elements of order 3 are either involutory centers or lie on involutory axes.

So the non-trivial elements of order prime to 7 have their fixed points on the 7 involutory axes. If a point P fixed by an element of order 7 is incident to an involutory axis, then $\mathcal{G}_P$ is divisible by $2p$; by looking at the list of subgroups in Result 2.3, $\mathcal{G}_P = \mathcal{G}$, which contradicts the fact that no point is fixed by $\mathcal{G}$. So no point fixed by an element of order 7 is incident to an involutory axis. Let S be a Sylow-7 group and let $x \in \mathcal{N}_{\mathcal{G}}(S)$ be of order 3. Since the points fixed by x lie on involutory axes, x can fix no points of $\mathcal{F}(S)$, and so acts fixed-point-freely on $\mathcal{F}(S)$. In particular, no point is fixed by $\mathcal{N}_{\mathcal{G}}(S)$.

What are the possiblities for $\mathcal{F}(S)$? S may fix no point of the plane, or it may be triangular, planar, or a generalized perspectivity. If S is a non-triangular generalized perspectivity, then there is a center $C(S)$ which is fixed by $\mathcal{N}_{\mathcal{G}}(S)$. This contradiction proves that S is not a non-triangular generalized perspectivity. $\mathcal{F}(S)$ is not a Baer subplane, since if it were, there would be an involutory axis $a(i)$ incident to a point P fixed by S. So if S is planar, the order m of the subplane satisfies Theorem 2.3, and $m^2 + m \leq n$. If S is triangular, the number of points fixed by S equals 3 which is less than n, so in either case the number of points fixed by S is less than or equal to $n + 1$.

There are $n + 1 - 3$ points on an involutory axis which are not involutory centers. Each of these points is incident to exactly one involutory axis since involutory axes meet in involutory centers. So there are $7(n - 2)$ of these points and 7 centers. In addition there are at most $8(n + 1)$ points fixed by the Sylow-7 groups. Thus $n^2 + n + 1 \leq 7(n - 2) + 7 + 8(n + 1)$, $n^2 - 14n \leq 0$, and so $n \leq 14$.

Now it follows that S is not planar of order m, since m has to satisfy $3|(m^2+m+1)$ and $m^2 + m \leq 14$. Thus $\mathcal{F}(S)$ is empty or is a triangle. No two Sylow-7 groups share a common fixed point P, since otherwise $\mathcal{G}_P$ contains two Sylow-7 groups which generate $\mathcal{G}$. Thus there will be either 0 or $8 \cdot 3$ points fixed by the Sylow-7 groups. Using this now to get an exact count of the number of points in Π gives that $n^2 + n + 1 = 7(n-2) + 7 + 0$ which has as solution $n = 4, 2$, or that $n^2 + n + 1 = 7(n-2) + 7 + 24$ which has solution $n = 8$. The planes of order 2, 4, and 8 are Desarguesian (page 144 of Dembowski [2]). The Desarguesian planes of order 2, 4 and 8 do admit $PSL(2,7)$ as a totally irregular collineation group.

$\square$

CHAPTER 6
CONCLUSION

In this thesis, a plane Π has been partially constructed in terms of the group $PSL(2,q)$, where $PSL(2,q)$ acts totally irregularly and contains an involutory homology. There is an upper bound of $\frac{3}{2}q^2$ on the order of the plane, and, except in a few cases, a lower one of q. It has been determined that if the plane exists, the element of order p is planar, or is a generalized elation, or acts fixed-point-freely. In the case where the element of order p acts fixed-point-freely, equations for the order n of Π have been found in terms of q. So far, the only value of q found which satisfies these equations is 7. The Desarguesian planes of order 9 and 11 have been characterized by $PSL(2,7)$ in this way. Number theoretical results may provide a way to further limit the values of q and n.

If the element of order p is a generalized elation, then, except possibly when $p = 3$, Π is either a Desarguesian plane of order q, or Π is a non-Desarguesian plane of order q^2 with a Desarguesian subplane of order q left invariant by $PSL(2,q)$. This second case allows the possibility of a new plane. The plane is not Desarguesian, and it seems likely that a similar argument as that which ruled out the Desarguesian plane will show the plane is not Hughes. Figueroa planes do not have order q^2. By Theorem 2.4, if $PSL(2,q)$ acts totally irregularly on a plane and contains an involutory homology, then $PSL(2,q)$ does not fix a point or a line; the only known planes which have a collineation group which does not fix a point and a line of the plane are the Desarguesian, Hughes, and Figueroa planes. So the plane of order q^2, if it exists, may be a new one.

The case where $p = 3$ and the element of order p is a generalized elation is still open. In this case, the elements of a Sylow p-group share a common center, and the subgroups of order p in S have distinct axes.

Finally, there is the case where the element of order p is planar. It has been shown that $\mathcal{N}_{\mathcal{G}}(S)$, for S a Sylow p-group, fixes at most one point and one line. It has been shown that if $\mathcal{N}_{\mathcal{G}}(S)$ fixes a line, then the element of order p is not planar, and that if $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point, the element of order p is not planar when q is prime. There remain the less restricted cases where either $\mathcal{N}_{\mathcal{G}}(S)$ fixes a point and q is not prime, or $\mathcal{N}_{\mathcal{G}}(S)$ fixes no point and no line. Some partial results have been found, although they have not been reported here.

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BIOGRAPHICAL SKETCH

Kristine Griffin was born in Abilene, Texas, on August 27, 1959. After moving to several states in her early years, her family settled in Florida. She attended University of Florida, receiving a B.S. in mathematics in the summer of 1981. She started graduate school in mathematics at the University of Florida in the fall of 1982, receiving a M.S. in 1984. During her time in graduate school she has been a teaching assistant, receiving a training in teaching through experience, as well as a more formal training in mathematics. She started her research study in finite geometry, and then expanded this study by studying a topic in finite groups and finite projective planes.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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